We’ve seen that the fact that Schwarzschild time coordinate $t$ and the proper time $\tau$ are not the same leads to the gravitational redshift. The relation between the wavelength of a photon at two distances $r_R$ and $r_E$ from a mass $M$ is

$$\frac{\lambda_R}{\lambda_E} = \sqrt{1 - \frac{2GM}{r_R}} / \sqrt{1 - \frac{2GM}{r_E}}$$  \hspace{1cm} (1)$$

We can derive the same formula from the photon’s four-momentum. In an observer’s locally flat frame, the energy of any object (including a photon) is

$$E_o = -o_t \cdot p$$  \hspace{1cm} (2)$$

since $o_t = [1, 0, 0, 0]$ in that local frame, and $p^t = E_o$. Using the expressions for these two four-vectors in the global Schwarzschild frame, we get

$$o'_t = \left[ \left( 1 - \frac{2GM}{r} \right)^{-1/2}, 0, 0, 0 \right]$$  \hspace{1cm} (3)$$

$$p'^t = E_\infty \left( 1 - \frac{2GM}{r} \right)^{-1}$$  \hspace{1cm} (4)$$

where $E_\infty$ is the energy of the photon as measured by an observer at infinity. The scalar product gives
That is, the energy of a photon increases the closer it gets to the mass. In terms of the wavelength, we can use the relation from quantum mechanics: 

\[ E = h\nu = \frac{hc}{\lambda} = \frac{h}{\lambda} \]

(taking \( c = 1 \) as usual). Thus

\[ \lambda_o = \lambda_\infty \sqrt{1 - \frac{2GM}{r}} \]  

or, if we take the ratio of the wavelengths at two finite radii as at the start:

\[ \frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1 - \frac{2GM}{r_R}}{1 - \frac{2GM}{r_E}}} \]  

which is the same as the original formula.

Another way of saying this is that an observer far from a mass sees light red-shifted compared to an observer near the mass or, conversely, the near observer sees incoming light blue-shifted compared to an observer far from the mass.