LOCAL FLAT FRAME FOR A CIRCULAR ORBIT

Here's another example of calculating a flat local reference frame in a global Schwarzschild metric. We'll follow the same procedure as in the case of the observer falling radially inwards. This time, we’re interested in an observer in a circular orbit of radius \( r \). First, we need to work out the four-velocity in Schwarzschild coordinates. We can get this from the conserved quantities \( e \) and \( l \).

\[
\frac{r^2 \, d\phi}{d\tau} = l \tag{1}
\]

\[
\left( 1 - \frac{2GM}{r} \right) \frac{dt}{d\tau} = e \tag{2}
\]

For a circular orbit, we can get \( l \) and \( e \) using our earlier formulas:

\[
l^2 = \frac{r^2 GM}{r - 3GM} \tag{3}
\]

\[
e = \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{1/2} \tag{4}
\]

Therefore

\[
\frac{d\phi}{d\tau} = \frac{l}{r^2} \tag{5}
\]

\[
= \frac{1}{r} \sqrt{\frac{GM}{r - 3GM}} \tag{6}
\]

\[
\frac{dt}{d\tau} = \frac{e}{1 - 2GM/r} \tag{7}
\]

\[
= \left( 1 - \frac{2GM}{r} \right)^{-1} \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{1/2} \tag{8}
\]
Since there is no motion in the radial or $\theta$ directions, the four-velocity is

$$o_t = u = \left[ \left( 1 - \frac{2GM}{r} \right)^{-1} \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{1/2} \right], 0, 0, \frac{1}{r} \sqrt{\frac{GM}{r - 3GM}}$$

(9)

This can be simplified a bit by using some algebra. The first component is

$$\left( 1 - \frac{2GM}{r} \right)^{-1} \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{1/2} = \left[ \frac{r - 3GM - GM + 4 (GM)^2}{(r - 3GM) (1 - 2GM/r)^2} \right]^{1/2}$$

(10)

$$= \left[ \frac{r (1 - 4GM/r + 4 (GM/r)^2)}{(r - 3GM) (1 - 2GM/r)^2} \right]^{1/2}$$

(11)

$$= \left[ \frac{r (1 - 2GM/r)^2}{(r - 3GM) (1 - 2GM/r)^2} \right]^{1/2}$$

(12)

$$= \frac{1}{\sqrt{1 - \frac{3GM}{r}}}$$

(13)

We can therefore rewrite \(^\text{9}\) as

$$o_t = u = \frac{1}{\sqrt{1 - \frac{3GM}{r}}} \left[ 1, 0, 0, \sqrt{\frac{GM}{r^3}} \right]$$

(14)

As a check, we can verify that $u \cdot u = -1$ (this is most easily done with Maple, but if you’re persistent it can be done by hand).

Now we align the $x$, $y$ and $z$ local directions with the global $\phi$, $-\theta$ and $r$ directions respectively. For $o_y$ and $o_z$, this means they will have the same global components as for a stationary observer, so that

$$o_y = \left[ 0, 0, -\frac{1}{r}, 0 \right]$$

(15)

$$o_z = \left[ 0, \sqrt{1 - \frac{2GM}{r}}, 0, 0 \right]$$

(16)
To find \( \mathbf{o}_x \), we start with \( \mathbf{o}_x \cdot \mathbf{o}_t = 0 \), which gives us

\[
g_{tt} \dot{\mathbf{o}}_t \dot{\mathbf{o}}_t^t + g_{\phi \phi} \dot{\mathbf{o}}_t \dot{\mathbf{o}}_x^\phi = 0 \tag{17}
\]

\[
- \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{1/2} \mathbf{o}_x^t + r \sqrt{\frac{GM}{r - 3GM}} \mathbf{o}_x^\phi = 0 \tag{18}
\]

\[
\left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{-1/2} r \sqrt{\frac{GM}{r - 3GM}} \mathbf{o}_x^\phi = \mathbf{o}_x^t \tag{19}
\]

Now we can use normalization, so that \( \mathbf{o}_x \cdot \mathbf{o}_x = 1 \). We get

\[
\left\{ 1 - \left( 1 - \frac{2GM}{r} \right) \frac{GM}{r - 3GM} \left[ 1 - \frac{GM}{r} \left( 1 - \frac{3GM}{r} \right)^{-1} \left( 1 - \frac{4GM}{r} \right) \right]^{-1} \right\} r^2 \left( \mathbf{o}_x^\phi \right)^2 = 1 \tag{20}
\]

The algebra is tedious so is best handled with Maple, and after simplifying, we get

\[
\mathbf{o}_x^\phi = \frac{1}{r} \sqrt{\frac{r - 2GM}{r - 3GM}} \tag{21}
\]

We can substitute this back into the earlier equation to get \( \mathbf{o}_x^t \), and, after simplifying, we get

\[
\mathbf{o}_x = \left[ \sqrt{\frac{rGM}{(r - 2GM)(r - 3GM)}}, 0, 0, \frac{1}{r} \sqrt{\frac{r - 2GM}{r - 3GM}} \right] \tag{22}
\]

The formula breaks down for \( r \leq 3GM \), which is presumably because if a photon approaches more closely than that, it spirals into \( r = 0 \). In particular, for \( r \leq 3GM \), it is not possible for \( dr/dt \) to be zero, which we assumed in this derivation.