EVENT HORIZON: PROPER TIME TO FALL INTO R = 0

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Continuing our exploration of the Schwarzschild metric and its behaviour at $r = 2GM$ (known as the event horizon), we’ll look here at the proper time an object measures as it crosses the event horizon. The metric is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

(1)

Consider an object starting at rest at a radial coordinate $r = R$ and falling radially in towards the event horizon. To work out its proper time, we start with the equation of motion for $r$:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\ell^2 + GM\left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) = \frac{1}{2}\left(e^2 - 1\right)$$

(2)

Since the object is at rest initially, $dr/d\tau = 0$ and its angular momentum is $\ell = 0$ so we get for its energy

$$e = \sqrt{1 - \frac{2GM}{R}}$$

(3)

The energy $e$ is a conserved quantity, so the equation of motion becomes

$$\frac{dr}{d\tau} = \sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R}}$$

(4)

We can integrate this to find the proper time that elapses as the object falls from $r = R$ to $r = 0$:

$$\int_{R}^{0} \frac{-dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \sqrt{2GM}\Delta\tau$$

(5)

(the minus sign in the integrand accounts for the fact that $r$ is decreasing).
Using Maple, the integral is
\[
\int \frac{-dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \sqrt{R} \left[ 2\sqrt{r(R-r)} + R \arctan \left( \frac{R-2r}{2\sqrt{r(R-r)}} \right) \right]
\] (6)

Plugging in the limits, we get

\[
\sqrt{2GM} \Delta \tau = \frac{\pi R^{3/2}}{2}
\] (7)

\[
\Delta \tau = \frac{\pi R^{3/2}}{\sqrt{8GM}}
\] (8)

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