PAINLEVÉ-GULLSTRAND COORDINATES: DERIVATION USING A LOCAL FLAT FRAME

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Earlier, we worked out the basis vectors in a locally flat frame for a freely falling observer near a black hole. These basis vectors are worked out by considering the four-velocity in two frames: the local, flat frame, and the Schwarzschild (S) frame. In particular, in the flat frame, \( u = o_t = [1, 0, 0, 0] \) so in the other frame, the four-velocity is the transformed time basis vector: \( u' = o_t' \). Using this argument, we worked out \( o_t' \) in the S frame for a freely falling observer and got

\[
o_t' = \left[ \left(1 - \frac{2GM}{r}\right)^{-1}, -\frac{\sqrt{2GM}}{r}, 0, 0 \right]
\]

(1)

In the flat frame, we can write the interval between two events as \( ds = [d\tau, dx, dy, dz] \). In the Painlevé-Gullstrand system, the time coordinate \( t \) is just the proper time of a freely falling observer, so \( dt = d\tau \). Still in the flat frame, we have therefore

\[
dt = -\eta_{ij} o_t^i ds^j = -o_t \cdot ds
\]

(2)

since only the component \( o_t^t \) is non-zero, and \( \eta_{tt} = -1 \) in flat space. Since this is a scalar product, it has the same value in any coordinate system, such as the S system where we have

\[
dt = o_t' \cdot ds' = g_{ij} o_t'^i ds'^j
\]

(3)

(4)

In the S system, we have

\[
ds' = [dt, dr, d\theta, d\phi]
\]

(5)

so
\[ dt = - \left[ - \left( 1 - \frac{2GM}{r} \right) \right] \left( \frac{2GM}{r} \right)^{-1} dt - \left( 1 - \frac{2GM}{r} \right)^{-1} \left( -\sqrt{\frac{2GM}{r}} \right) dr \]

(6)

\[ = dt + \left( 1 - \frac{2GM}{r} \right)^{-1} \sqrt{\frac{2GM}{r}} dr \]

(7)

This agrees with the earlier result for Painlevé-Gullstrand.