CHRISTOFFEL SYMBOLS: SYMMETRY

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Reference: Moore, Thomas A., A General Relativity Workbook, University Science Books (2013) - Chapter 17; Box 17.3.
Post date: 22 Dec 2013.

The Christoffel symbols are defined in terms of the basis vectors in a given coordinate system as:

\[ \frac{\partial e_i}{\partial x^j} = \Gamma^k_{ij} e_k \]  

(1)

Remember that the basis vectors \( e_i \) are defined so that

\[ ds^2 = ds \cdot ds \]  

(2)

\[ = (dx^i e_i) \cdot (dx^j e_j) \]  

(3)

\[ = e_i \cdot e_j dx^i dx^j \]  

(4)

\[ \equiv g_{ij} dx^i dx^j \]  

(5)

In a locally flat frame using rectangular spatial coordinates, the basis vectors \( e_i \) are all constants, so from \( [1] \) all the Christoffel symbols must be zero: \( \Gamma^k_{ij} = 0 \).

Now let’s look at the second covariant derivative of a scalar field \( \Phi \):

\[ \nabla_i \nabla_j \Phi = \nabla_i (\partial_j \Phi) \]  

(6)

\[ = \partial_i \partial_j \Phi - \Gamma^k_{ij} \partial_k \Phi \]  

(7)

where in \([6]\) we used rule 1 for the covariant derivative: the covariant derivative of a scalar is the same as the ordinary derivative.

In the locally flat frame, this equation reduces to

\[ \nabla_i \nabla_j \Phi = \partial_i \partial_j \Phi \]  

(8)

Since the covariant derivative is a tensor, this is a tensor equation, and since ordinary partial derivatives commute, this equation is the same if we swap the indices \( i \) and \( j \). Tensor equations must have the same form in all coordinate systems, so this implies that \([7]\) must also be invariant if we
swap $i$ and $j$. This means that the Christoffel symbols are symmetric under exchange of their two lower indices:

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

(9)

At first glance, this seems wrong, since from the definition this symmetry implies that

$$\frac{\partial e_i}{\partial x^j} = \frac{\partial e_j}{\partial x^i}$$

(10)

In 2-D polar coordinates, if we take the usual unit vectors $\hat{r}$ and $\hat{\theta}$ then both these vectors are constants as we change $r$ and both of them change when we change $\theta$, so it’s certainly not true that $\partial \hat{r}/\partial \theta = \partial \hat{\theta}/\partial r$, for example. However, remember that the basis vectors we’re using are not the usual unit vectors; rather they are defined so that condition is true. In polar coordinates, we have

$$ds^2 = dr^2 + r^2 d\theta^2$$

(11)

so

$$e_r = \hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$$

(12)

$$e_\theta = r \hat{\theta} = -r \sin \theta \hat{x} + r \cos \theta \hat{y}$$

(13)

For the derivatives, we have

$$\frac{\partial e_r}{\partial \theta} = -\sin \theta \hat{x} + \cos \theta \hat{y} = \hat{\theta}$$

(14)

$$\frac{\partial e_\theta}{\partial r} = -\sin \theta \hat{x} + \cos \theta \hat{y} = \hat{\theta}$$

(15)

Thus the condition is actually satisfied here.

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