RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

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We’ll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

\[
R^i_{\ j\ell m} \equiv \partial_\ell \Gamma^i_{\ mj} - \partial_m \Gamma^i_{\ \ell j} + \Gamma^j_{\ \ell k} \Gamma^i_{\ km} - \Gamma^j_{\ \ell k} \Gamma^i_{\ km}
\]  

As usual, we need the Christoffel symbols, but we’ve already worked these out.

\[
\Gamma^t_{\ ij} = \begin{bmatrix} 0 & GM (1 - \frac{2GM}{r})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]  

\[
\Gamma^r_{\ ij} = \begin{bmatrix} \frac{GM}{r^2} (1 - \frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & \frac{-GM}{r^2} (1 - \frac{2GM}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r (1 - \frac{2GM}{r}) & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta (1 - \frac{2GM}{r}) \end{bmatrix}
\]  

\[
\Gamma^\theta_{\ ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \sin \theta \cos \theta \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]  

\[
\Gamma^\phi_{\ ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} \cot \theta & 0 & 0 \end{bmatrix}
\]

We can plug these into the formula above to get $R^t_{\ rtr}$. We have

\[
R^t_{\ rtr} = \partial_t \Gamma^t_{\ r\ell} - \partial_r \Gamma^t_{\ tr} + \Gamma^k_{\ r\ell} \Gamma^t_{\ \ell k} - \Gamma^k_{\ tr} \Gamma^t_{\ rk}
\]
We can work out these terms one at a time (only the index $k$ is summed):

$$\partial_t \Gamma^t_{rr} = 0$$  \hspace{2cm} (7)

$$-\partial_r \Gamma^t_{tr} = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-2} \left(\frac{2GM}{r^2}\right)$$  \hspace{2cm} (8)

$$\Gamma^k_{rr} \Gamma^t_{kt} = \Gamma^r_{rr} \Gamma^t_{rt}$$  \hspace{2cm} (10)

$$\Gamma^k_{rr} \Gamma^t_{rk} = -\left(\Gamma^t_{rt}\right)^2$$  \hspace{2cm} (12)

Adding these up we get

$$R^t_{tr} = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1}$$  \hspace{2cm} (14)

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \to \infty$, $R^t_{tr} \to 0$ so the further we get from the mass, the less curved the spacetime becomes.