As an example of the Riemann tensor in 2-d curved space we can use our old standby of the surface of a sphere. As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

\[ \frac{d}{d\tau} \left( g_{aj} \dot{x}^j \right) - \frac{1}{2} \partial_a g_{ij} \dot{x}^i \dot{x}^j = 0 \]  

(1)

\[ \ddot{x}^m + \Gamma^m_{ij} \dot{x}^i \dot{x}^j = 0 \]  

(2)

where as usual a dot denotes a derivative with respect to proper time \( \tau \).

For a sphere, the interval is

\[ ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

(3)

Note that \( r \) (the radius of the sphere) is a constant here. From (1) we get, with \( a = \theta \):

\[ r^2 \ddot{x}^j - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \]  

(4)

Dividing through by \( r^2 \) and comparing with (2) we get

\[ \Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta \]  

(5)

\[ \Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 0 \]  

(6)

With \( a = \phi \) we have

\[ 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0 \]  

(7)

\[ 2 \cot \theta \dot{\theta} \dot{\phi} + \ddot{\phi} = 0 \]  

(8)

\[ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot \theta \]  

(9)

\[ \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0 \]  

(10)
RIEMANN TENSOR FOR SURFACE OF A SPHERE

We can use these results to get the Riemann tensor. Unfortunately, in the form $R^a_{\ bcd}$, the Riemann tensor doesn’t have all the symmetries of the form $R_{abcd}$, so if we want the latter form, we need to work out the former form first and then use

\[ R_{abcd} = g_{af} R^f_{bced} \]

\[ = g_{af} \left( \partial_c \Gamma^f_{db} - \partial_d \Gamma^f_{cb} + \Gamma^k_{db} \Gamma^f_{ck} - \Gamma^k_{cb} \Gamma^f_{kd} \right) \]

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

\[ R_{\theta\phi\theta\phi} = g_{\theta f} R^f_{\phi\theta\phi} \]

\[ \quad = g_{\theta \theta} R^0_{\phi \theta \phi} \]

\[ \quad = r^2 \left( \partial_\theta \Gamma^\theta_{\phi \phi} - \partial_\phi \Gamma^\phi_{\theta \phi} + \Gamma^k_{\phi \phi} \Gamma^\phi_{\theta k} - \Gamma^k_{\theta \phi} \Gamma^\phi_{k \phi} \right) \]

\[ \quad = r^2 \left( \sin^2 \theta - \cos^2 \theta - 0 + 0 + \cos^2 \theta \right) \]

\[ \quad = r^2 \sin^2 \theta \]

\[ R_{\theta\phi\theta\theta} = g_{\theta \theta} R^\theta_{\phi \theta \theta} \]

\[ \quad = r^2 \left( \partial_\theta \Gamma^\theta_{\phi \theta} - \partial_\phi \Gamma^\phi_{\theta \theta} + \Gamma^k_{\phi \theta} \Gamma^\phi_{\theta k} - \Gamma^k_{\theta \phi} \Gamma^\phi_{k \theta} \right) \]

\[ \quad = -R_{\theta\phi\theta\phi} \]

\[ \quad = -r^2 \sin^2 \theta \]

\[ R_{\phi\theta\theta\phi} = g_{\phi \phi} R^\phi_{\theta \theta \phi} \]

\[ \quad = r^2 \sin^2 \theta \left( \partial_\phi \Gamma^\phi_{\theta \theta} - \partial_\theta \Gamma^\theta_{\phi \phi} + \Gamma^k_{\phi \phi} \Gamma^\phi_{\theta k} - \Gamma^k_{\phi \theta} \Gamma^\phi_{k \phi} \right) \]

\[ \quad = r^2 \sin^2 \theta \left( - \frac{1}{\sin^2 \theta} - 0 + 0 + \frac{\cos^2 \theta}{\sin^2 \theta} \right) \]

\[ \quad = -r^2 \sin^2 \theta \]

\[ R_{\phi\theta\phi\theta} = g_{\phi \phi} R^\phi_{\theta \phi \theta} \]

\[ \quad = r^2 \sin^2 \theta \left( \partial_\phi \Gamma^\phi_{\theta \phi} - \partial_\theta \Gamma^\theta_{\phi \phi} + \Gamma^k_{\phi \phi} \Gamma^\phi_{\theta k} - \Gamma^k_{\phi \theta} \Gamma^\phi_{k \phi} \right) \]

\[ \quad = -R_{\phi\theta\phi\theta} \]

\[ \quad = r^2 \sin^2 \theta \]

Finally, we can calculate one of the other components to verify that it’s zero.
Riemann Tensor for Surface of a Sphere

\[ R_{\theta\theta\theta\theta} = g_{\theta\theta} R_{\theta\theta}^\theta \] (30)

\[ = r^2 \left( \partial_\theta \Gamma_{\theta\theta}^\theta - \partial_\theta \Gamma_{\theta\theta}^\theta + \Gamma_{\theta\theta}^k \Gamma_{\theta\theta}^\theta - \Gamma_{\theta\theta}^\theta \Gamma_{\theta\theta}^\theta \right) \] (31)

\[ = 0 \] (32)

Pingbacks

Pingback: Riemann tensor for 3-d spherical coordinates