RIEMANN TENSOR FOR 3-D SPHERICAL COORDINATES

We saw that the Riemann tensor for the surface of a sphere had a non-zero component, indicating that this is a curved space. If we use spherical coordinates in 3-d space, however, the Riemann tensor should be zero, since this is a flat space.

As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

\[
\frac{d}{d\tau} \left( g_{aj} \dot{x}^j \right) - \frac{1}{2} \partial_a g_{ij} \dot{x}^i \dot{x}^j = 0 \quad (1)
\]
\[
\ddot{x}^m + \Gamma^m_{ij} \dot{x}^j \dot{x}^i = 0 \quad (2)
\]

where as usual a dot denotes a derivative with respect to proper time \( \tau \).

For spherical coordinates, the interval is

\[
ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (3)
\]

Note that \( r \) is now a variable, rather than the constant radius of the sphere in the 2-d system.

From (1) we get, with \( a = \theta \):

\[
2r \dot{r} \dot{\theta} + r^2 \ddot{\theta} - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \quad (4)
\]

Dividing through by \( r^2 \) and comparing with (2) we get

\[
\Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta \quad (5)
\]
\[
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \quad (6)
\]
\[
\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\theta} = 0 \quad (7)
\]

With \( a = \phi \) we have
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\[ 2r \sin^2 \theta \dot{r} \dot{\phi} + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} \] = 0 \quad (8)

\[ \frac{2}{r} \dot{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \ddot{\phi} = 0 \quad (9) \]

\[ \Gamma_{\theta \phi \phi} = \Gamma_{\phi \theta \phi} = \cot \theta \quad (10) \]

\[ \Gamma_{r \phi \phi} = \Gamma_{\phi r \phi} = \frac{1}{r} \quad (11) \]

\[ \Gamma_{\theta \phi \theta} = \Gamma_{\phi \theta \phi} = 0 \quad (12) \]

For \( a = r \), we have

\[ \ddot{r} - \frac{1}{2} \left( 2r \dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2 \right) = 0 \quad (13) \]

\[ \ddot{r} - r \dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 = 0 \quad (14) \]

\[ \Gamma_{\theta \theta} = -r \quad (15) \]

\[ \Gamma_{\phi \phi} = -r \sin^2 \theta \quad (16) \]

We can use these results to get the Riemann tensor.

\[ R_{abcd} = g_{af} R_{f b c d} \quad (17) \]

\[ = g_{af} \left( \partial_c \Gamma_{db}^{f} - \partial_d \Gamma_{cb}^{f} + \Gamma_{ck}^{f} \Gamma_{db}^{k} - \Gamma_{db}^{k} \Gamma_{ck}^{f} \right) \quad (18) \]

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

\[ R_{\theta \phi \theta \phi} = g_{\theta f} R_{\phi \theta \phi}^{f} \quad (19) \]

\[ = g_{\theta \theta} R_{\phi \phi}^{\theta} \quad (20) \]

\[ = r^2 \left( \partial_{\theta} \Gamma_{\phi \phi}^{\theta} - \partial_{\phi} \Gamma_{\theta \phi}^{\theta} + \Gamma_{\phi \phi}^{k} \Gamma_{\theta \phi}^{k} - \Gamma_{\theta \phi}^{k} \Gamma_{\phi \phi}^{k} \right) \quad (21) \]

\[ = r^2 \left( \sin^2 \theta - \cos^2 \theta - 0 - \frac{r \sin^2 \theta}{r} + \cos^2 \theta \right) \quad (22) \]

\[ = 0 \quad (23) \]

The other components of the Riemann tensor can be evaluated in the same way, and they all come out to zero, so spherical coordinates represent flat 3-d space.