A natural way of defining force as a four-vector is as the derivative with respect to proper time $\tau$ of the four-momentum, that is

$$F^i = \left(\frac{dp}{d\tau}\right)^i$$  \hspace{1cm} (1)$$

We can relate this to the stress-energy tensor by using the fact that $T^{ij}$ is the rate of flow of $p^i$ in direction $j$ (when $j$ is a spatial coordinate). If $i = t$, $T^{tj}$ is the rate of flow of energy in the $j$ direction, while if $i$ is a spatial coordinate, $T^{ij}$ is the rate of flow of that momentum component in the $j$ direction. If we want to know the rate at which component $i$ of four-momentum flows across a small planar patch of area $A$, we can take the unit normal to the area $n$. If $n$ is a four-vector then since it’s a purely spatial direction, its time component is zero and we have

$$n = [0, n^x, n^y, n^z]$$ \hspace{1cm} (2)$$

Just as the component of an ordinary vector along a particular direction is given by the scalar product, we can think of one row in the stress-energy tensor as an analogue of a four-vector. For example, if we take the second row we get the components

$$T_x = [T^{xt}, T^{xx}, T^{xy}, T^{xz}]$$ \hspace{1cm} (3)$$

The first component $T^{xt} = T^{tx}$ is the rate of flow of energy in the $x$ direction, while the last three components give the rate of flow of $x$ momentum as a 3-d vector. The rate of flow of $x$ momentum in the $n$ direction is then

$$\left(\frac{dp}{d\tau}\right)^x = A n \cdot T_x$$ \hspace{1cm} (4)$$

$$= A g_{ij} n^i T^{xj}$$ \hspace{1cm} (5)$$

where we’ve multiplied by $A$ since the components $T^{xj}$ give the rate of flow
per unit area. Notice this formula works because \( n^t = 0 \), so only the three components of spatial momentum contribute to this product. The same formula applies to the \( y \) and \( z \) components as well.

What about \( \left( \frac{dp}{d\tau} \right)^t \)? If we apply the same idea, we have

\[
T_t = [T^{tt}, T^{tx}, T^{ty}, T^{tz}] \tag{6}
\]

and

\[
\left( \frac{dp}{d\tau} \right)^t = A g_{ij} n^i T^{tj} \tag{7}
\]

This is the rate of energy flow along the direction \( n \). Thus the general formula for the rate of four-momentum flow is

\[
\left( \frac{dp}{d\tau} \right)^i = A g_{kj} n^k T^{ij} \tag{8}
\]

In the general case where the fluid’s four-velocity is \( u^i \), the stress-energy tensor is

\[
T^{ij} = (\rho_0 + P_0) u^i u^j + P_0 g^{ij} \tag{9}
\]

where \( \rho_0 \) is the energy density of the fluid and \( P_0 \) is the pressure, both measured in the fluid’s rest frame. The momentum flow in the fluid’s rest frame is

\[
\left( \frac{dp}{d\tau} \right)^i = A g_{kj} n^k \left[ (\rho_0 + P_0) u^i u^j + P_0 g^{ij} \right] \tag{10}
\]

However, in the fluid’s rest frame, the fluid’s own four-velocity is \( u^i = [1, 0, 0, 0] \) so combined with [2] we have

\[
u \cdot n = g_{kj} u^j n^k = 0 \tag{11}
\]

Since this is a scalar, it has the same value in all coordinate systems, so the first term in [10] is zero in every coordinate system, so we get

\[
\left( \frac{dp}{d\tau} \right)^i = A g_{kj} n^k P_0 g^{ij} \tag{12}
\]

\[= A g_{kj} n^k P_0 \tag{13}\]

\[= A n^i P_0 \tag{14}\]
If now we place a real wall at the patch $A$, and this wall absorbs all the momentum that flows into it, then $\frac{dp}{d\tau}$ is the force on that wall. The magnitude of the force is

$$
F = \sqrt{\frac{dp}{d\tau} \cdot \frac{dp}{d\tau}} \tag{15}
$$

$$
= AP_0 \sqrt{g_{ij}n^i n^j} \tag{16}
$$

$$
= AP_0 \tag{17}
$$

where the result follows because $n$ is a unit vector. Again, this result is a scalar so it applies in all coordinate systems.