DOMINANT ENERGY CONDITION

The dominant energy condition (DEC) states that if \( a^i \) is any four-vector that is causal, that is, it satisfies the conditions

\[
\begin{align*}
    a \cdot a &\leq 0 \\
    a^t &> 0
\end{align*}
\]

then we require the stress-energy tensor \( T^{ij} \) to satisfy the condition that if

\[
b^j = -T^{ij} g_{jk} a^k
\]

then \( b \) is also a causal four-vector. The causal condition is just a way of saying that a four-vector is either timelike (if \( a \cdot a < 0 \)) or lightlike (if \( a \cdot a = 0 \)). The DEC is a condition on the stress-energy tensor which amounts to saying that taking the scalar product of one of its rows or columns with a causal vector cannot produce a non-causal (spacelike) vector. Physically, this says that nothing can move faster than light. Note that it’s not a property that is automatically true of any stress-energy tensor; rather it is a condition imposed on the tensor to make it physically realistic.

We can use the DEC to show that the momentum density of a perfect fluid is always causal. The tensor in the fluid’s rest frame is

\[
T^{ij} = \begin{bmatrix}
\rho & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & P
\end{bmatrix}
\]

The momentum density is defined as the first row (or column) of the tensor:

\[
\pi^i \equiv T^{di}
\]
so $\pi$ is causal in this frame. In a local orthonormal frame (LOF) the tensor’s components are

$$T^p_{\,\,\ obs} = \eta^{ij} \eta_{km} (o_j)^k \eta^{pq} \eta_{rs} (o_q)^r \, T^{ms}$$  (9)

where $o_i$ are the orthonormal basis vectors in the LOF. If we plug in the definition (5) we get

$$\pi^p_{\,\,\ obs} = T^p_{\,\,\ obs} = \eta^{ij} \eta_{km} (o_j)^k \eta^{pq} \eta_{rs} (o_q)^r \, T^{ms}$$  (10)

$$= \eta^{ij} \eta_{km} (o_j)^k \eta^{pq} \eta_{rs} (o_q)^r \, T^{ms}$$  (11)

$$= \left[ -T^{ms} \eta_{km} (o_t)^k \right] \eta^{pq} \eta_{rs} (o_q)^r$$  (12)

where we got the last line by using the fact that $\eta^{ij}$ is diagonal and $\eta^{tt} = -1$. The term in square brackets looks like 3 as long as $o_t$ is a causal vector. However, this vector is just the observer’s four-velocity $u_{\ obs}$ measured in the fluid’s frame, so

$$u_{\ obs} \cdot u_{\ obs} = -1$$  (13)

$$u_{\ obs}^t = \gamma > 0$$  (14)

Thus $o_t$ is indeed causal, so we can invoke the DEC to say that if we define a vector $B^s$ by

$$B^s \equiv -T^{ms} \eta_{km} (o_t)^k$$  (15)

then $B^s$ must be causal. We then get

$$\pi^p_{\,\,\ obs} = B^s \eta^{pq} \eta_{rs} (o_q)^r$$  (16)

$$= \eta^{pq} [B^s \eta_{rs} (o_q)^r]$$  (17)

$$= \eta^{pq} B_{\ obs, q}$$  (18)

$$= B^p_{\ obs}$$  (19)
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With this definition, we can calculate

$$\pi_{obs} \cdot \pi_{obs} = B_{obs} \cdot B_{obs} \quad (20)$$

However, we know that \( B \) is causal because that’s how we defined it in [15] and since its magnitude is a scalar, it is the same in all coordinate systems, so we must have \( \pi_{obs} \cdot \pi_{obs} = B_{obs} \cdot B_{obs} \leq 0 \). As for showing that \( \pi_{obs}^t > 0 \), we can observe that

$$\pi_{obs}^t = \eta^{tq} [B^s \eta_{rs} (o_q)^r] \quad (21)$$
$$\pi_{obs}^t = -B^s \eta_{rs} (o_t)^r \quad (22)$$

Since \( o_t = \gamma [1, v_x, v_y, v_z] \) we see that \( \pi_{obs}^t \) is the Lorentz transformation of the causal vector \( B^s \) and a Lorentz transformation never changes the spacetime nature of a four-vector (that is, timelike remains timelike, etc) so since \( B \) is causal, \( B^t > 0 \) and therefore \( \pi_{obs}^t > 0 \) as well. This condition is known as the weak energy condition or WEC.

Another property of the stress-energy tensor that can be derived from the DEC is as follows. In the rest frame of a perfect fluid, [4] holds, so the DEC condition [3] says, for some arbitrary causal vector \( a \) we get another causal vector \( b \):

$$b^i = -T_{ij} g_{jk} a^k \quad (23)$$
$$b^t = \rho a^t \quad (24)$$
$$b^m = -P a^m \text{ for } m = x, y, z \quad (25)$$

Since \( b \) is causal, we must have for all choices of \( a \):

$$b \cdot b = -\rho^2 (a^t)^2 + P^2 \sum_{m=x,y,z} (a^m)^2 \leq 0 \quad (26)$$

Because \( a \) is causal, we have

$$ (a^t)^2 \geq \sum_{m=x,y,z} (a^m)^2 \quad (27)$$

The constraint on \( \rho \) and \( P \) in [26] comes in the case of equality in [27], in which case we have

$$-\rho^2 + P^2 \leq 0 \quad (28)$$

and since \( \rho > 0 \) this amounts to
Finally, we can revisit the case of the stress-energy tensor that gives negative pressure, $T^{ij} = -\Lambda g^{ij}$, where $\Lambda$ is a positive scalar. In this case, if we apply the DEC to some causal vector $a$ we get

$$b^i = -T^{ij}g_{jk}a^k$$  \hspace{1cm} (30)

$$= \Lambda g^{ij}g_{jk}a^k$$  \hspace{1cm} (31)

$$= \Lambda a^i$$  \hspace{1cm} (32)

Since $b$ is a positive scalar multiplied by a causal vector, it too must be causal, so this stress-energy tensor satisfies the DEC.