EINSTEIN TENSOR OF ZERO IMPLIES A ZERO RICCI TENSOR

We wish to prove that \( G_{ij} = 0 \) if and only if \( R_{ij} = 0 \). The Einstein tensor is defined as

\[
G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R
\]  

Clearly if the Ricci tensor \( R_{ij} = 0 \) then \( G_{ij} = 0 \) (since the curvature scalar is the contraction of the Ricci tensor: \( R = g_{ij} R^{ij} \)). To prove the converse, suppose \( G_{ij} = 0 \). Then we can multiply both sides by \( g_{ij} \) to get

\[
0 = g_{ij} R^{ij} - \frac{1}{2} g_{ij} g^{ij} R
\]

\[
= R - 2R
\]

\[
R = 0
\]

Since \( g_{ij} g^{ij} = 4 \). With \( G_{ij} = 0 \) and \( R = 0 \), \( 1 \) tells us that \( R^{ij} = 0 \). QED.