GRAVITOMAGNETIC ACCELERATION NEAR A ROTATING STAR

As we’ll (hopefully) see later, the perturbations to the weak field metric around a rotating spherical star are

\[ h_{tt} = h_{xx} = h_{yy} = h_{zz} = \frac{2GM}{r} \]  
\[ h_{tx} = h_{xt} = \frac{2GSy}{r^3} \]  
\[ h_{ty} = h_{yt} = -\frac{2GSx}{r^3} \]

with all other perturbations being zero. Here \( S \) is the star’s angular momentum (assumed to be pointing in the +z direction) and

\[ r = \sqrt{x^2 + y^2 + z^2} \]

The gravitomagnetic matrix is

\[ F_{kj} = \partial_k h_{tj} - \partial_j h_{tk} \]

so to calculate it for the rotating star, we need a few derivatives. First,

\[ \partial_x \left( \frac{1}{r^3} \right) = -3 \frac{1}{r^4} \partial_x r \]  
\[ = -3 \frac{x}{r^5} \]  
\[ \partial_y \left( \frac{1}{r^3} \right) = -3 \frac{y}{r^5} \]  
\[ \partial_z \left( \frac{1}{r^3} \right) = -3 \frac{z}{r^5} \]

Since \( F_{kj} = -F_{jk} \), \( F_{ii} = 0 \) and we need only the off-diagonal elements
Therefore

\[ \partial_x h_{ty} = \frac{6GSx^2}{r^5} - \frac{2GS}{r^3} \]  
(10)

\[ = \frac{2GS}{r^5} (3x^2 - (x^2 + y^2 + z^2)) \]  
(11)

\[ = \frac{2GS}{r^5} (2x^2 - y^2 - z^2) \]  
(12)

\[ \partial_y h_{tx} = \frac{2GS}{r^5} (x^2 - 2y^2 + z^2) \]  
(13)

\[ \partial_z h_{tx} = -\frac{6GSyz}{r^5} \]  
(14)

\[ \partial_z h_{ty} = \frac{6GSxz}{r^5} \]  
(15)

Putting it together we get

\[
F_{kj} = \begin{bmatrix}
0 & \partial_x h_{ty} - \partial_y h_{tx} & -\partial_z h_{tx} \\
-(\partial_x h_{ty} - \partial_y h_{tx}) & 0 & -\partial_z h_{ty} \\
\partial_z h_{tx} & \partial_z h_{ty} & 0
\end{bmatrix}
\]  
(16)

\[
= \frac{2GS}{r^5} \begin{bmatrix}
0 & x^2 + y^2 - 2z^2 & 3yz \\
-x^2 - y^2 + 2z^2 & 0 & -3xz \\
3yz & 3xz & 0
\end{bmatrix}
\]  
(17)

For a particle on the \( x \) axis moving in the \( +x \) direction at speed \( v \ll 1 \) (the latter assumption is necessary for the above equations to be valid), we have \( y = z = 0 \) so

\[
F_{kj} = \frac{2GS}{x^3} \begin{bmatrix}
0 & x^2 & 0 \\
-x^2 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(18)

\[
= \frac{2GS}{x^3} \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(19)

so the gravitomagnetic acceleration is, with \( v^j = [v, 0, 0] = v\delta^j_1 \)

\[
\eta^{ik} F_{kj} v^j = \delta_i^1 F_{kj} \delta^j_1 v = F_{i1} v = -\frac{2GSv}{x^3} [0, 1, 0]
\]  
(20)

(21)

(22)
Thus the gravitomagnetic acceleration is in the $y$ direction towards the $x$ axis, perpendicular to the velocity.