We can now have a look at the equation of motion for a particle moving in the weak field, slow velocity limit. We begin with the geodesic equation

\[ \ddot{x}^m + \Gamma^m_{ij} \dot{x}^i \dot{x}^j = 0 \quad (1) \]

where a dot indicates a derivative with respect to proper time \( \tau \). Using the chain rule we can write the first term as

\[ \frac{d^2 x^m}{d\tau^2} = \frac{d}{d\tau} \left( \frac{dx^m}{d\tau} \right) \]
\[ = \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dx^m}{d\tau} \right) \]
\[ = \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{d}{d\tau} \frac{dx^m}{dt} \right) \]
\[ = u_t \frac{d}{d\tau} \frac{d^2 x^m}{dt^2} + v^m \frac{dt}{d\tau} \frac{d}{dt} \frac{dx^m}{dt} \]
\[ = u_t u_t \frac{d^2 x^m}{dt^2} + v^m \frac{dt}{d\tau} \frac{d}{dt} \]
\[ = u_t u_t \frac{d^2 x^m}{dt^2} + v^m \frac{du^t}{d\tau} \quad (7) \]

where

\[ v^m \equiv \frac{dx^m}{dt} \quad (8) \]

Now suppose the particle is moving slowly so that both \( u^i \) (where \( i \) is a spatial index) and \( v^t \) are small, so we can ignore second order terms. In this approximation, \( u^t \approx 1 \) so we have from (1) with \( m = t \)
GRAVITOELECTRIC AND GRAVITOMAGNETIC DENSITIES

\[ \frac{du^t}{d\tau} = -\Gamma^q_{ij} \dot{x}^j \dot{x}^i \]  
(9)
\[ = -\Gamma^q_{ij} u^j u^i \]  
(10)
\[ \approx -\Gamma^q_{tt} u^t u^t \]  
(11)
\[ = -\Gamma^q_{tt} \]  
(12)

Therefore

\[ v^m \frac{du^t}{d\tau} \approx -v^m \Gamma^q_{tt} \]  
(13)

The Christoffel symbols in terms of the metric are

\[ \Gamma^m_{ij} = \frac{1}{2} g^{ml} \left( \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right) \]  
(14)

For a stationary field, all time derivatives are zero, so

\[ \Gamma^q_{tt} = -\frac{1}{2} g^{tl} \partial_t g_{tt} \]  
(15)

In the weak field limit, we are using a metric that is perturbed from the flat space metric:

\[ g_{ij} = \eta_{ij} + h_{ij} \]  
(16)
\[ g^{ij} = \eta^{ij} - h^{ij} \]  
(17)

So to first order in \( h_{ij} \):

\[ \Gamma^q_{tt} = -\frac{1}{2} \left( \eta^{tl} - h^{tl} \right) \partial_t (\eta_{tt} + h_{tt}) \]  
(18)
\[ = -\frac{1}{2} \eta^{tt} \partial_t h_{tt} + \frac{1}{2} h^{tt} \partial_t \eta_{tt} \]  
(19)
\[ = 0 \]  
(20)

since all time derivatives are zero in the stationary field, and all derivatives of the flat space metric are zero as well. Therefore in this approximation it becomes

\[ \frac{d^2 x^m}{d\tau^2} = u^t u^t \frac{d^2 x^m}{dt^2} = \frac{d^2 x^m}{dt^2} \]  
(21)

and the geodesic equation becomes
\[
\frac{d^2 x^m}{dt^2} = -\Gamma_{ij}^m u^j u^i \quad (22)
\]

where to get the last line we used the symmetry of the Christoffel symbols: \(\Gamma_{it}^m = \Gamma_{ti}^m\).

In the weak field limit, the Christoffel symbols become

\[
\Gamma_{ij}^m = \frac{1}{2} \eta^{ml} \left( \partial_j h_{il} + \partial_i h_{lj} - \partial_l h_{ji} \right) \quad (25)
\]

In this approximation

\[
\Gamma_{it}^m = \frac{1}{2} \eta^{ml} \left( \partial_t h_{it} + \partial_l h_{lt} - \partial_l h_{lt} \right) \quad (26)
\]

\[
= -\frac{1}{2} \eta^{ml} \partial_t h_{lt} \quad (27)
\]

since time derivatives are zero for a stationary field.

Also

\[
2\Gamma_{it}^m u^i = u^i \eta^{ml} \left( \partial_t h_{it} + \partial_l h_{lt} - \partial_l h_{lt} \right) \quad (28)
\]

\[
= u^i \eta^{ml} \left( \partial_l h_{lt} - \partial_l h_{lt} \right) \quad (29)
\]

Putting it all together, the geodesic equation \(24\) becomes

\[
\frac{d^2 x^m}{dt^2} = \eta^{ml} \left[ \frac{1}{2} \partial_t h_{tt} + u^i \left( \partial_t h_{it} - \partial_t h_{lt} \right) \right] \quad (30)
\]

Remember that all indices (except \(t\)) range over only spatial components.

In the weak field limit, the perturbations to the metric are given by

\[
h_{jm} = 2G \int \frac{2T_{jm} - \eta_{jm} T}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}' \quad (31)
\]

The numerator in the integrand is analogous to charge density in electromagnetism. The first term in \(30\) results from the density

\[
\rho_g = 2T_{tt} - \eta_{tt} T \quad (32)
\]

which is called the gravitoelectric energy density, since it gives rise to an acceleration of the particle that does not depend on the particle’s velocity.
in a similar way to an electric field accelerating a charge independently of
the charge’s velocity.

The second term in (30) results from the density

$$\Pi_j \equiv T_{tj} - \frac{1}{2} \eta_{tj} T$$  \hspace{1cm} (33)

which is called the gravitomagnetic current density since it contributes to an
acceleration that is proportional to the particle’s velocity, in a similar way
to the Lorentz force arising from magnetism.

PINGBACKS

Pingback: Gravitelectric and gravitomagnetic densities for the vacuum
Pingback: Gravitomagnetic acceleration is perpendicular to velocity
Pingback: Gravitomagnetic acceleration near a rotating star
Pingback: Gravitelectric and gravitomagnetic acceleration for a moving wire
Pingback: Gravitelectric and gravitomagnetic acceleration for parallel plates