PROBABILITY CURRENT

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Using the Schrödinger equation we can derive an interesting quantity called the *probability current*. Using the probabilistic interpretation of the wave function, the probability of a particle being between $x=a$ and $x=b$ is

$$P_{ab}=\int_a^b|\mathbf{\Psi}^*\mathbf{\Psi}|\,dx \tag{1}$$

The rate of change of this probability can then be expressed in terms of spatial derivatives using the Schrödinger equation:

$$\begin{aligned}
\frac{dP_{ab}}{dt} &= \int_a^b\left[\frac{\partial\mathbf{\Psi}^*}{\partial t}\mathbf{\Psi}+\frac{\partial\mathbf{\Psi}}{\partial t}\mathbf{\Psi}^*\right]dx && (2)\\
&= \int_a^b\left\{-\frac{i\hbar}{2m}\frac{\partial^2\mathbf{\Psi}^*}{\partial x^2}-\frac{1}{i\hbar}V\mathbf{\Psi}^*\right\}\mathbf{\Psi}dx && (3)\\
&\quad+\int_a^b\left\{\frac{i\hbar}{2m}\frac{\partial^2\mathbf{\Psi}}{\partial x^2}+\frac{1}{i\hbar}V\mathbf{\Psi}\right\}\mathbf{\Psi}^*dx && (4)\\
&= \frac{i\hbar}{2m}\int_a^b\left[\frac{\partial^2\mathbf{\Psi}}{\partial x^2}\mathbf{\Psi}^*-\frac{\partial^2\mathbf{\Psi}^*}{\partial x^2}\mathbf{\Psi}\right]dx && (5)
\end{aligned}$$

We can now apply integration by parts to each term.

$$\begin{aligned}
\int_a^b\frac{\partial^2\mathbf{\Psi}}{\partial x^2}\mathbf{\Psi}^*dx &= \left.\frac{\partial\mathbf{\Psi}}{\partial x}\mathbf{\Psi}^*\right|_a^b-\int_a^b\frac{\partial\mathbf{\Psi}}{\partial x}\frac{\partial\mathbf{\Psi}^*}{\partial x}dx && (6)\\
-\int_a^b\frac{\partial^2\mathbf{\Psi}^*}{\partial x^2}\mathbf{\Psi}dx &= -\left.\frac{\partial\mathbf{\Psi}^*}{\partial x}\mathbf{\Psi}\right|_a^b+\int_a^b\frac{\partial\mathbf{\Psi}}{\partial x}\frac{\partial\mathbf{\Psi}^*}{\partial x}dx && (7)
\end{aligned}$$

Adding these terms together, we get

$$\frac{dP_{ab}}{dt}=\frac{i\hbar}{2m}\left[\left.\frac{\partial\mathbf{\Psi}}{\partial x}\mathbf{\Psi}^*\right|_a^b-\left.\frac{\partial\mathbf{\Psi}^*}{\partial x}\mathbf{\Psi}\right|_a^b\right] \tag{8}$$

If we define the *probability current* as

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \frac{\partial\Psi}{\partial x}\Psi^*\right) \tag{9}$$

we can write the rate of change of probability as

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t) \tag{10}$$

We can interpret this as a probability conservation law. The rate of change of probability in the spatial interval $x \in [a,b]$ is the difference between the probability current flowing in at one end ($x = a$, say) and out at the other ($x = b$).

As an example, if the wave function is given by

$$\Psi(x,t) = \left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{11}$$

(we've taken the constant in front so that it normalizes the wave function), then

$$\frac{\partial\Psi}{\partial x} = -\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{12}$$

So for the probability current we get

$$J(x,t) = -\frac{i\hbar}{2m}\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)-it\right]}\left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{13}$$

$$+\frac{i\hbar}{2m}\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)+it\right]}\left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)-it\right]} \tag{14}$$

$$= 0 \tag{15}$$

A bit of an anti-climax after all that mathematics.

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