The number of microstates in an Einstein solid with $N$ oscillators and $q$ energy quanta is

$$\Omega = \binom{q + N - 1}{q}$$  \hspace{1cm} (1)$$

For any macroscopic solid, both $q$ and $N$ are large numbers (on the order of Avogadro’s number, or $10^{23}$) so the factorials in $\Omega$ are very large numbers, not calculable on most computers. To get estimates of $\Omega$ we can use Stirling’s approximation for the factorials. The derivation of this approximation for the high temperature case $q \gg N$ (lots more energy quanta than oscillators) is given in Schroeder’s book, so I’ll deal here with the low temperature case $q \ll N$.

Writing out the binomial coefficient

$$\binom{q + N - 1}{q} = \frac{(q + N - 1)!}{q!(N - 1)!}$$  \hspace{1cm} (2)$$

$$\approx \frac{(q + N)!}{q!N!}$$  \hspace{1cm} (3)$$

where we’ve the fact that if we multiply a very large number like $(q + N - 1)!$ by a merely large number like $N$, the original very large number is essentially unchanged.

We can now take logs and use Stirling’s approximation for the log of a factorial

$$\ln n! \approx n \ln n - n$$  \hspace{1cm} (4)$$

We get

$$\ln \Omega \approx (q + N) \ln (q + N) - q - N - q \ln q + q - N \ln N + N$$  \hspace{1cm} (5)$$

$$= (q + N) \ln (q + N) - q \ln q - N \ln N$$  \hspace{1cm} (6)$$

If we now make the assumption that $q \ll N$, we get
\[ \ln \Omega \approx (q + N) \ln \left( N \left(1 + \frac{q}{N}\right) \right) - q \ln q - N \ln N \]  
(7)

\[ = (q + N) \left[ \ln N + \ln \left(1 + \frac{q}{N}\right) \right] - q \ln q - N \ln N \]  
(8)

\[ \approx q \ln N + (q + N) \frac{q}{N} - q \ln q \]  
(9)

\[ = q \ln \frac{N}{q} + q + \frac{q^2}{N} \]  
(10)

\[ \approx q \ln \frac{N}{q} + q \]  
(11)

where to get the third line we’ve used the approximation \( \ln (1 + x) \approx x \) for \(|x| \ll 1\), and in the last line we’ve neglected the \( \frac{q^2}{N} \) term in the \( q \ll N \) limit. Exponentiating this result gives the approximate value for \( \Omega \):

\[ \Omega \approx \left( \frac{Ne}{q} \right)^q \]  
(12)

[The corresponding result in the high temperature case is \( \Omega \approx (qe/N)^N \) which could have been predicted easily, since \( q \) and \( N \) appear symmetrically in the approximation \(3\).]

This result applies also to the two-state paramagnet with \( N \) magnetic dipoles and \( N_\uparrow \) energy quanta, since the system is formally equivalent to an Einstein solid (we’re distributing the energy quanta among dipoles rather than oscillators). The multiplicity of the paramagnet is then

\[ \Omega \approx \left( \frac{Ne}{N_\uparrow} \right)^{N_\uparrow} \]  
(13)

Finally, we can use Stirling’s approximation on \(2\) directly to get an approximation for the case where \( N \) and \( q \) are any large numbers, without one necessarily being much larger than the other. We have

\[ \Omega = \binom{q + N - 1}{q} = \frac{(q + N - 1)!}{q!(N - 1)!} \]  
(14)

\[ = \frac{1}{q!} \frac{N (q + N)!}{N! (q + N)} \]  
(15)

\[ = \frac{N (q + N)!}{q + N \cdot q!N!} \]  
(16)

Stirling’s approximation for a large factorial is
\[ n! \approx \sqrt{2\pi nn^ne^{-n}} \]  

so we get

\[
\Omega \approx \frac{N}{q+N} \sqrt{\frac{2\pi (q+N)}{q+N}} \frac{(q+N)^q+N^q}{2\pi q N q^N N^N e^{-(q+N)}}
\]

\[ = \sqrt{\frac{N}{2\pi q (q+N)}} \left( \frac{q+N}{q} \right)^q \left( \frac{q+N}{N} \right)^N \]

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