ORTHONORMAL BASIS AND ORTHOGONAL COMPLEMENT

Once we have defined an inner product defined on a vector space $V$, we can create an orthonormal basis for $V$. A list of vectors $(e_1, e_2, \ldots, e_n)$ is orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij} \quad (1)$$

That is, any pair of vectors is orthogonal, and all the vectors have norm 1. In 3-d space, the unit vectors along the three axes form an orthonormal list.

Given an orthonormal list, we can construct a vector from the vectors in that list by

$$v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n \quad (2)$$

$$= \sum_{i=1}^{n} a_i e_i \quad (3)$$

for $a_i \in \mathbb{F}$. The norm of $v$ has a simple form:

$$\langle v, v \rangle^2 = \left\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i \right\rangle \quad (4)$$

$$= \sum_{i=1}^{n} \langle a_i e_i, a_i e_i \rangle + \text{zero terms} \quad (5)$$

$$= \sum_{i=1}^{n} |a_i|^2 \quad (6)$$

The 'zero terms' in the second line are terms involving $\langle a_i e_i, a_j e_j \rangle$ for $i \neq j$ which are all zero because of (1).

This result shows that an orthonormal list of vectors is linearly independent, since if we form the linear combination
Then \( \langle v, v \rangle = 0 \) so from \( 6 \) we must have all \( a_i = 0 \), which means the list is linearly independent.

If we have an orthonormal list \( (e_1, e_2, \ldots, e_n) \) that is also a basis for \( V \), then any vector \( v \in V \) can be written as

\[
v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n
\]

(8)

The coefficients \( a_i \) can be found by taking the inner product \( \langle e_i, v \rangle = a_i \) (using \( 1 \)), so we have

\[
v = \sum_{i=1}^{n} \langle e_i, v \rangle e_i
\]

(9)

For example, in 3-d space, the 3 unit vectors along the \( x, y, z \) axes form an orthonormal basis for the space. However, the unit vectors along the \( x, y \) axes form an orthonormal list, but this is not a basis for 3-d space since no vector with a \( z \) component can be written as a linear combination of these two vectors.

If we have any basis (not necessarily orthonormal), we can form an orthonormal basis using the Gram-Schmidt orthogonalization procedure. We’ve already met this in the context of quantum mechanics, and the derivation for a general finite vector space is much the same, so I’ll just quote the result. The procedure is iterative and follows these steps:

The first vector \( e_1 \) in the orthonormal basis is defined by

\[
e_1 = \frac{v_1}{|v_1|}
\]

(10)

where \( v_1 \) is the first vector (well, any vector, really) in the non-orthonormal basis.

Given vector \( e_{j-1} \) in the orthonormal basis, we can form \( e_j \) from the formula

\[
e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{|v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i|}
\]

(11)

\( e_j \) clearly has norm 1, and we can check that \( \langle e_i, e_j \rangle = \delta_{ij} \) by direct calculation. Note that although we’ve indexed the vectors \( v_i \) in the original basis, we can take them in any order when calculating the orthonormal basis via the Gram-Schmidt procedure.
Orthogonal complement. Suppose we have a subset $U$ (not necessarily a subspace) of $V$. Then we can define the orthogonal complement $U^\perp$ of $U$ as the set of all vectors that are orthogonal to all vectors $u \in U$. More formally:

$$U^\perp \equiv \{ v \in V | \langle v, u \rangle = 0 \text{ for all } u \in U \}$$  \hspace{1cm} (12)

A useful general theorem is as follows.

**Theorem 1.** If $U$ is a subspace of $V$, then $V = U \oplus U^\perp$. (Recall the direct sum)

**Proof.** Given an orthonormal basis of $U$: $(e_1, e_2, \ldots, e_n)$, we can write any $v \in V$ as the sum

$$v = \sum_{i=1}^{n} \langle e_i, v \rangle e_i + v - \sum_{i=1}^{n} \langle e_i, v \rangle e_i$$ \hspace{1cm} (13)

On the RHS, we’ve just added and subtracted the same term from $v$. Since the first term is a linear combination of the basis vectors of $U$, the overall sum is a vector in $U$. To see that the second term is in $U^\perp$, take the inner product with any of the basis vectors $e_k$:

$$\langle e_k, v - \sum_{i=1}^{n} \langle e_i, v \rangle e_i \rangle = \langle e_k, v \rangle - \sum_{i=1}^{n} \langle e_i, v \rangle \delta_{ik}$$ \hspace{1cm} (14)

$$= \langle e_k, v \rangle - \sum_{i=1}^{n} \langle e_i, v \rangle$$ \hspace{1cm} (15)

$$= \langle e_k, v \rangle - \langle e_k, v \rangle$$ \hspace{1cm} (16)

$$= 0$$ \hspace{1cm} (17)

Finally, since the two vector spaces in a direct sum can have only the zero vector in their intersection, we need to show that $U \cap U^\perp = \{0\}$. However if a vector $v$ is in both $U$ and $U^\perp$ then it must be orthogonal to itself, so $\langle v, v \rangle = 0$ which implies $v = 0$. \hfill \Box

Thus any vector space $V$ can be decomposed into two orthogonal subspaces (assuming that $V$ has any subspaces other than $\{0\}$).

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