UNITARY OPERATORS

Another important type of operator is the unitary operator $U$, which is defined by the condition that it is surjective and that

$$|Uu| = |u|$$

for all $u \in V$. That is, a unitary operator preserves the norm of all vectors. The identity matrix $I$ is a special case of a unitary operator, as it doesn’t change any vector, but multiplying $I$ by any complex number $\alpha$ with $|\alpha| = 1$ also preserves the norm, so $\alpha I$ is another unitary operator.

Because $U$ preserves the norm of all vectors, the only vector that can be in the null space of $U$ is the zero vector, meaning that $U$ is also injective. As it is both injective and surjective, it is invertible.

**Theorem 1.** For a unitary operator $U$, $U^\dagger = U^{-1}$.

**Proof.** From its definition and the properties of an adjoint operator, we have

$$|Uu|^2 = \langle Uu, Uu \rangle = \langle u, U^\dagger Uu \rangle = \langle u, u \rangle$$

Therefore, $U^\dagger U = I$ so $U^\dagger = U^{-1}$. □

**Theorem 2.** Unitary operators preserve inner products, meaning that $\langle Uu, Uv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

**Proof.** Since $U^\dagger = U^{-1}$ we have

$$\langle Uu, Uv \rangle = \langle u, U^\dagger Uv \rangle = \langle u, v \rangle$$

□

**Theorem 3.** Acting on an orthonormal basis $(e_1, \ldots, e_n)$ with a unitary operator $U$ produces another orthonormal basis.
**Proof.** Suppose the orthonormal basis is converted to another set of vectors \((f_1, \ldots, f_n)\) by \(U\):

\[
f_i = U e_i
\]  

(6)

Then

\[
\langle f_i, f_j \rangle = \langle U e_i, U e_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}
\]  

(7)

Thus \((f_1, \ldots, f_n)\) are an orthonormal set. Since the orthonormal basis \((e_1, \ldots, e_n)\) spans \(V\) (by assumption) and the set \((f_1, \ldots, f_n)\) contains \(n\) linearly independent orthonormal vectors, \((f_1, \ldots, f_n)\) is also an orthonormal basis for \(V\). □

**Theorem 4.** If one orthonormal basis \((e_1, \ldots, e_n)\) is converted to another \((f_1, \ldots, f_n)\) by a unitary operator \(U\), then the matrix elements of \(U\) are the same in both bases.

**Proof.** This is just a special case of the more general theorem that states that any operator that transforms one set of basis vectors into another has the same matrix elements in both bases. In this case, the proof is especially simple:

\[
U_{ki}(\{e\}) = \langle e_k, U e_i \rangle = \langle U^{-1} f_k, f_i \rangle = \langle U^\dagger f_k, f_i \rangle = \langle f_k, U f_i \rangle = U_{ki}(\{f\})
\]  

(8)  

(9)  

(10)  

(11)  

(12) □

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