TRANSLATION INVARIANCE IN TWO DIMENSIONS

In preparation for an examination of rotation invariance, we’ll have a look at translational invariance in two dimensions. We can apply much of what we did with translation in one dimension, where we showed that the momentum $P$ is the generator of translations. In particular, the translation operator $T(\varepsilon)$ for an infinitesimal translation $\varepsilon$ is

$$T(\varepsilon) = I - \frac{i\varepsilon}{\hbar}P$$  \hspace{1cm} (1)

In two dimensions, we can write an infinitesimal translation as $\delta a$ where

$$\delta a = \delta a_x \hat{x} + \delta a_y \hat{y}$$  \hspace{1cm} (2)

In one dimension, we showed earlier that

$$\langle x | T(\varepsilon) | \psi \rangle = \psi(x - \varepsilon)$$  \hspace{1cm} (3)

The analogous relation in two dimensions is

$$\langle x, y | T(\delta a) | \psi \rangle = \psi(x - \delta a_x, y - \delta a_y)$$  \hspace{1cm} (4)

We can verify that the correct form for $T(\delta a)$ is

$$T(\delta a) = I - \frac{i}{\hbar} \delta a \cdot \mathbf{P}$$  \hspace{1cm} (5)

$$= I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y)$$  \hspace{1cm} (6)

Using the representation of momentum in the position basis, which is

$$P_x = -i\hbar \frac{\partial}{\partial x}$$  \hspace{1cm} (7)

$$P_y = -i\hbar \frac{\partial}{\partial y}$$  \hspace{1cm} (8)

the LHS of (4) is, using $\langle x, y | \psi \rangle = \psi(x, y)$:
\langle x, y \mid T (\delta a) \mid \psi \rangle = \left\langle x, y \mid I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y) \mid \psi \right\rangle
\end{array} \right) (9)
\end{equation}

\begin{equation}
= \psi (x, y) - \delta a_x \frac{\partial \psi}{\partial x} - \delta a_y \frac{\partial \psi}{\partial y} (10)
\end{equation}

The last line is also what we get if we expand the RHS of $4$ to first order in $\delta a$, which verifies that $5$ is correct, so that the two-dimensional momentum $P$ is the generator of two-dimensional translations.

We can apply the exponentiation technique we used in the one-dimensional case to obtain the translation operator for a finite translation in two dimensions. We need to be careful that we don’t run into problems with non-commuting operators, but in view of $7$ and $8$ and the fact that derivatives with respect to different independent variables commute, we see that

\begin{equation}
[ P_x, P_y ] = 0 \tag{11}
\end{equation}

We can divide a finite translation $a$ into $N$ small steps, each of size $\frac{a}{N}$, so that the translation is

\begin{equation}
T (a) = \left( I - \frac{i}{\hbar N} a \cdot P \right)^N \tag{12}
\end{equation}

Because the two components of momentum commute, we can take the limit of this expression to get the exponential form:

\begin{equation}
T (a) = \lim_{N \to \infty} \left( I - \frac{i}{\hbar N} a \cdot P \right)^N = e^{-i P / \hbar} \tag{13}
\end{equation}

Again, because the two components of momentum commute, we can combine two translations, by $a$ and then by $b$, to get

\begin{equation}
T (b) T (a) = e^{-i b / \hbar} e^{-i a / \hbar} = e^{-i (a + b) / \hbar} = T (b + a) \tag{14}
\end{equation}

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