ROTATIONS THROUGH A FINITE ANGLE; USE OF POLAR COORDINATES

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Chapter 12, Exercise 12.2.3.
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The angular momentum operator $L_z$ is the generator of rotations in the $xy$ plane. We did the derivation for infinitesimal rotations, but we can generalize this to finite rotations in a similar manner to that used for translations.

The unitary transformation for an infinitesimal rotation is

$$U[R(\varepsilon z \hat{z})] = I - \frac{i\varepsilon z L_z}{\hbar} \tag{1}$$

For rotation through a finite angle $\phi_0$, we divide up the angle into $N$ small angles, so $\varepsilon z = \phi_0 / N$. Rotation through the full angle $\phi_0$ is then given by

$$U[R(\phi_0 \hat{z})] = \lim_{N \to \infty} \left( I - \frac{i\phi_0 L_z}{N\hbar} \right)^N = e^{-i\phi_0 L_z / \hbar} \tag{2}$$

The limit follows because the only non-trivial operator involved is $L_z$, so no commutation problems arise.

In rectangular coordinates, $L_z$ has the relatively non-obvious form

$$L_z = X P_y - Y P_x \tag{3}$$

$$= -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \tag{4}$$

so it’s not immediately clear that $2$ does in fact lead to the desired rotation. Trying to calculate the exponential with $L_z$ expressed this way is not easy, given that the two terms $x \frac{\partial}{\partial y}$ and $y \frac{\partial}{\partial x}$ don’t commute.

It turns out that $L_z$ has a much simpler form in polar coordinates, and there are two ways of converting it to polar form. First, we recall the transformation equations.
\[ x = \rho \cos \phi \]  
\[ y = \rho \sin \phi \]  
\[ \rho = \sqrt{x^2 + y^2} \]  
\[ \phi = \tan^{-1} \frac{y}{x} \]  

From the chain rule, we can convert the derivatives:

\[ \frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \cos \phi}{\partial x} \frac{\partial}{\partial (\cos \phi)} \]  
\[ = \frac{\partial \rho}{\partial x} - \sin \phi \frac{\partial \phi}{\partial x} \left( -\sin \phi \right) \frac{\partial}{\partial \phi} \]  
\[ = \frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi} \]  

Using similar methods, we get for the other derivative

\[ \frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y} \frac{\partial}{\partial \rho} + \frac{\partial \sin \phi}{\partial y} \frac{\partial}{\partial (\sin \phi)} \]  
\[ = \frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi} \]  

Plugging these into (4) we have

\[ L_z = -i \hbar \left[ x \left( \frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi} \right) - y \left( \frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi} \right) \right] \]  
\[ = -i \hbar \frac{x^2 + y^2}{\rho^2} \frac{\partial}{\partial \phi} \]  
\[ = -i \hbar \frac{\partial}{\partial \phi} \]  

Another method of converting \( L_z \) to polar coordinates is to consider the effect of \( U[R] \) for an infinitesimal rotation \( \varepsilon_z \) on a state vector expressed in polar coordinates \( \psi(\rho, \phi) \). Shankar states that

\[ \langle \rho, \phi | U[R] | \psi(\rho, \phi) \rangle = \psi(\rho, \phi - \varepsilon_z) \]
If you don’t believe this, it can be shown using a method similar to that for the one-dimensional translation. In this case, we’re dealing with position eigenkets in polar coordinates, so we have

$$U[R] \vert \rho, \phi \rangle = \vert \rho, \phi + \varepsilon_z \rangle$$  \hspace{1cm} (19)

Applying this, we get

$$\vert \psi_{\varepsilon_z} \rangle = U[R] \vert \psi \rangle$$  \hspace{1cm} (20)

$$= U[R] \int_0^{2\pi} \int_0^\infty \langle \rho, \phi \vert \rho, \phi \rangle \rho d\rho d\phi$$  \hspace{1cm} (21)

$$= \int_0^{2\pi} \int_0^\infty \vert \rho, \phi + \varepsilon_z \rangle \langle \rho, \phi \vert \rho, \phi \rangle \rho d\rho d\phi$$  \hspace{1cm} (22)

$$= \int_0^{2\pi} \int_0^\infty \vert \rho', \phi' \rangle \langle \rho', \phi' - \varepsilon_z \vert \rho' d\rho' d\phi'$$  \hspace{1cm} (23)

where in the last line, we used the substitution $\phi' = \phi + \varepsilon_z$. (The substitution $\rho' = \rho$ is used just to give the radial variable a different name in the integrand.) We can use the same limits of integration for $\phi$ and $\phi'$, since we just need to ensure that the integral covers the total range of angles. It then follows that

$$\langle \rho, \phi \vert \psi_{\varepsilon_z} \rangle = \int_0^{2\pi} \int_0^\infty \delta (\rho - \rho') \delta (\phi - \phi') \langle \rho', \phi' - \varepsilon_z \vert \rho' d\rho' d\phi'$$  \hspace{1cm} (24)

$$= \psi (\rho, \phi - \varepsilon_z)$$  \hspace{1cm} (25)

Combining this with 1 we have

$$\langle \rho, \phi \vert I - \frac{i\varepsilon_z L_z}{\hbar} \vert \psi \rangle = \psi (\rho, \phi - \varepsilon_z)$$  \hspace{1cm} (27)

Expanding the RHS to order $\varepsilon_z$ we have

$$\langle \rho, \phi \vert I - \frac{i\varepsilon_z L_z}{\hbar} \vert \psi \rangle = \psi (\rho, \phi) - \varepsilon_z \frac{\partial \psi}{\partial \phi}$$  \hspace{1cm} (28)

from which 17 follows again.

Once we have $L_z$ in this form, the exponential form of a finite rotation is easier to interpret, for we have, from 2
\[ e^{-i \phi_0 L_z / \hbar} = \exp \left[ -\phi_0 \frac{\partial}{\partial \phi} \right] \]

\[ = 1 - \phi_0 \frac{\partial}{\partial \phi} + \frac{\phi_0^2}{2!} \frac{\partial^2}{\partial \phi^2} + \ldots \]  

(29)

(30)

Applying this to a state function \( \psi(\rho, \phi) \), we see that we get the Taylor series for \( \psi(\rho, \phi - \phi_0) \), so the exponential does indeed represent a rotation through a finite angle.