TWO-DIMENSIONAL HARMONIC OSCILLATOR – PART 2: SERIES SOLUTION

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Chapter 12, Exercise 12.3.7 (6) - (7).

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[If some equations are too small to read easily, use your browser’s magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In this post, we’ll continue with the solution of the 2-d isotropic harmonic oscillator. In the [last post], we started with the ODE for the radial function in the form

\[-\frac{\hbar^2}{2\mu} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} \right) R + \frac{1}{2} \mu \omega^2 \rho^2 R = ER\]  

(1)

We introduced dimensionless variables

\[y \equiv \sqrt{\frac{\mu \omega}{\hbar}} \rho\]  

(2)

\[\varepsilon \equiv \frac{E}{\hbar \omega}\]  

(3)

and found that \( R \) could be written as

\[R(y) = y^{|m|} e^{-y^2/2} U(y)\]  

(4)

with \( U \) given by the solution of the ODE

\[U'' + \left( \frac{2|m| + 1}{y} - 2y \right) U' + (2\varepsilon - 2|m| - 2) U = 0\]  

(5)

We can solve this by using a power series of the form

\[U(y) = \sum_{r=0}^{\infty} C_r y^r\]  

(6)

where the coefficients \( C_r \) are constants.
The derivatives are
\[ U' = \sum_{r=0}^{\infty} C_r r y^{r-1} \] (7)
\[ = 0 + C_1 + 2C_2 y + 3C_3 + y^2 + \ldots \] (8)
\[ = \sum_{r=0}^{\infty} C_{r+1} (r+1) y^r \] (9)
\[ U'' = \sum_{r=0}^{\infty} C_{r+1} r (r+1) y^{r-1} \] (10)
\[ = 0 + (1)(2) C_2 + (2)(3) C_3 y + \ldots \] (11)
\[ = \sum_{r=0}^{\infty} C_{r+2} (r+1) (r+2) y^r \] (12)

Plugging these into 5 we have (we’ll drop the absolute value signs on \(|m|\) to make the notation simpler; we can restore them at the end):

\[ \sum_{r=0}^{\infty} C_{r+2} (r+1) (r+2) y^r + (2m+1) \sum_{r=0}^{\infty} C_r y r^{r-2} - \ldots \] (13)
\[ = 2 \sum_{r=0}^{\infty} C_r r y r^r + 2 (\varepsilon - m - 1) \sum_{r=0}^{\infty} C_r y^r = 0 \] (14)

The second sum in the first line is

\[ \sum_{r=0}^{\infty} C_r r y r^{r-2} = 0 + C_1 y^{-1} + 2C_2 + 3C_3 y + \ldots \] (15)
\[ = \sum_{r=-1}^{\infty} C_{r+2} (r+2) y^r \] (16)

The sum thus becomes

\[ (2m+1) C_1 y^{-1} + \sum_{r=0}^{\infty} y^r C_{r+2} (r+2)^2 + 2 \sum_{r=0}^{\infty} y^r C_r [-r + \varepsilon - m - 1] = 0 \] (17)

A basic theorem about power series is that if the sum of the series equals zero for all \(y\), then the coefficient of each power must be zero. This shows that \(C_1 = 0\) since otherwise the series would blow up as \(y \to 0\). This results in a recursion relation for the \(C_r\):

\[ C_{r+2} = \frac{2(r + m + 1 - \varepsilon)}{(r+2)^2} C_r \] (18)
Since $C_1 = 0$, all $C_r = 0$ for odd $r$. For large $r$ we have

$$\frac{C_{r+2}}{C_r} \to \frac{2}{r}$$ \hspace{1cm} (19)

If the series is allowed to be infinite, this leads to a divergent series as we can see from the following (based on Shankar’s section 7.3). Suppose we look at $y^m e^{y^2}$, which clearly goes to infinity at large $y$ (remember, $m$ is positive). In series form this is

$$y^m e^{y^2} = \sum_{k=0}^{\infty} \frac{y^{2k+m}}{k!}$$ \hspace{1cm} (20)

The coefficient $C_n$ of $y^n$, with $n = 2k + m$ in this series is

$$C_n = \frac{1}{[(n-m)/2]!}$$ \hspace{1cm} (21)

Similarly,

$$C_{n+2} = \frac{1}{[(n+2-m)/2]!}$$ \hspace{1cm} (22)

The ratio is

$$\frac{C_{n+2}}{C_n} = \frac{[(n-m)/2]!}{[(n+2-m)/2]!} = \frac{1}{(n-m)/2 + 1} \to \frac{2}{n}$$ \hspace{1cm} (23)

In other words, the coefficients of our series solution have the same behaviour for large $r$ as those in the series for $y^m e^{y^2}$. Referring back to (4) we see that this gives an overall behaviour for the radial function $R$ of

$$R \to y^m e^{-y^2/2} y^m e^{y^2} = y^{2m} e^{y^2/2}$$ \hspace{1cm} (26)

Thus if we allow the series for $U$ to be infinite, the overall solution diverges, which is not acceptable. We therefore require that the series terminates at some finite value of $r$, and from (18) we see that this happens if

$$\varepsilon = r + m + 1$$ \hspace{1cm} (27)

for some $r$. From the definition this gives us the allowed values for the energy.
\[ E = \hbar \omega (r + |m| + 1) \quad (28) \]
\[ = \hbar \omega (2k + |m| + 1) \quad (29) \]

where the last line follows because \( r \) must be even. If

\[ n \equiv 2k + |m| \quad (30) \]

then the allowed energies are

\[ E = \hbar \omega (n + 1) \quad (31) \]

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