LEVI-CIVITA ANTISYMMETRIC TENSOR, VECTOR PRODUCTS AND SYSTEMS OF 3 FERMIONS

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Chapter 12, Exercise 12.4.1.
Post date: 16 May 2017

[If some equations are too small to read easily, use your browser’s magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The Levi-Civita symbol $\varepsilon_{ijk}$ is defined as $+1$ if $i,j,k$ have the values 1,2,3 (in that order), 2,3,1 or 3,1,2. Swapping any pair of indices multiplies the value by $-1$, so that, for example, $\varepsilon_{123} = +1$ and $\varepsilon_{213} = -1$. If two indices are the same, such as $i = j = 1$, then swapping them leaves $\varepsilon_{11k}$ unchanged so the requirement that $\varepsilon_{ijk} = -\varepsilon_{jik}$ means that $\varepsilon_{ijk} = 0$ if any two of its indices are equal.

The symbol is actually an antisymmetric tensor of rank 3, and is found frequently in physical and mathematical equations. One example is in the cross product of two 3-d vectors. If

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad (1)$$

we can work out the components of $\mathbf{c}$ in the usual way by calculating the determinant:

$$\mathbf{c} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (2)$$

$$= (a_2 b_3 - b_2 a_3) \hat{x}_1 - (a_1 b_3 - b_1 a_3) \hat{x}_2 + (a_1 b_2 - b_1 a_2) \hat{x}_3 \quad (3)$$

where I’ve used $\hat{x}_1 = \hat{x}$, $\hat{x}_2 = \hat{y}$ and $\hat{x}_3 = \hat{z}$.

Using $\varepsilon_{ijk}$ we can write this in the compact form

$$\mathbf{c} = \sum_{i,j,k} \varepsilon_{ijk} \hat{x}_i a_j b_k \quad (4)$$

as can be verified by expanding the sum and comparing with $\[3\]$.

The Levi-Civita symbol can be used to write a completely antisymmetric wave function for a set of three fermions. Suppose the wave function for a single fermion in state $n$ with coordinate $x_a$ is $U_n(x_a)$ (where both $n$ and $a$
can take values 1, 2 or 3). Then a completely antisymmetric wave function is

\[ \psi_A(x_1, x_2, x_3) = \frac{1}{\sqrt{6}} \varepsilon_{ijk} U_i(x_1) U_j(x_2) U_k(x_3) \]  

Equation 5

The factor of \( \frac{1}{\sqrt{6}} \) is for normalization and assumes that the \( U_n \) are all normalized wave functions.

Swapping the locations \( x_1 \) and \( x_2 \), for example, is equivalent to swapping \( i \) and \( j \) in the sum, which produces the negative of the original sum. That is

\[ \psi_A(x_2, x_1, x_3) = \frac{1}{\sqrt{6}} \varepsilon_{ijk} U_i(x_2) U_j(x_1) U_k(x_3) \]  

Equation 6

\[ = \frac{1}{\sqrt{6}} \varepsilon_{jik} U_i(x_1) U_j(x_2) U_k(x_3) \]  

Equation 7

\[ = - \frac{1}{\sqrt{6}} \varepsilon_{ijk} U_i(x_1) U_j(x_2) U_k(x_3) \]  

Equation 8

\[ = - \psi_A(x_1, x_2, x_3) \]  

Equation 9

The same argument applies to swapping the other pairs of locations.

PINGBACKS

Pingback: Vector operators; transformation under rotation
Pingback: Pauli matrices: properties
Pingback: Pauli matrices: A useful identity
Pingback: Electromagnetic Lorentz invariant
Pingback: Levi-civita tensor