ROTATION OF A VECTOR WAVE FUNCTION

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Chapter 12, Exercise 12.5.1.
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We’ve seen that for a rotation by an infinitesimal angle \( \varepsilon \) about the \( z \) axis, a scalar wave function transforms according to

\[
\psi(x, y) \rightarrow \psi(x + \varepsilon y, y - \varepsilon x)
\]  

(1)

The meaning of this transformation can be seen in the figure:

The physical system represented by the wave function \( \Psi \) is rigidly rotated by the angle \( \varepsilon \), so that the value of \( \Psi \) at point \( A \) is now sitting over the point \( B \). However, in the primed (rotated) coordinate system, the numerical value of the coordinates of the point \( B \) in the figure are the same as the numerical values that the point \( A \) had in the original, unrotated coordinates. That is

\[
(x'_B, y'_B) = (x_A, y_A)
\]  

(2)

Just as \( B \) is obtained from \( A \) by rotating \( A \) by \( +\varepsilon \), we can obtain \( A \) from \( B \) by rotating by \( -\varepsilon \). For any given point, the primed (rotated) and unprimed (unrotated) coordinates are related by (all relations are to first order in \( \varepsilon \)):
\[ x' = x - y\varepsilon_z \] (3)
\[ y' = y + x\varepsilon_z \] (4)

The inverse relations are obtained by a rotation by \(-\varepsilon_z\):

\[ x = x' + y'\varepsilon_z \] (5)
\[ y = y' - x'\varepsilon_z \] (6)

After rotation, the values of \(\Psi'\) are related to the values \(\Psi\) before rotation by rotating through the angle \(-\varepsilon_z\), so that

\[ \Psi'(x, y) = \Psi(x + y\varepsilon_z, y - x\varepsilon_z) \] (7)

Now suppose the wave function is a vector \(V = V_x\hat{x} + V_y\hat{y}\). The situation is as shown:

The initial unrotated vector \(V\) is the value of the wave function at point \(A\) (and is entirely in the \(x\) direction for convenience). After rotation, the vector gets moved to \(B\) and is also rotated so that it now makes an angle \(\varepsilon_z\) with the original \(x\) axis. However, its direction is now along the \(x'\) axis, which makes an angle of \(\varepsilon_z\) with the original \(x\) axis.

In this case, each component of \(V\) still gets transformed in the same way as the scalar function above, but the vector itself is also rotated. If the components \(V_x\) and \(V_y\) of the vector were constants, then the rotated vector is given by applying the 2-d rotation matrix

\[
R = \begin{bmatrix}
1 & -\varepsilon_z \\
\varepsilon_z & 1
\end{bmatrix}
\] (8)

so we get \(V' = RV\), or, in components:
If \( V_x \) and \( V_y \) vary from point to point, then we must apply the transformation \([\mathbf{1}]\) to each component, so that the overall transformation is

\[
\begin{align*}
V'_x &= V_x (x + \varepsilon_z y, y - \varepsilon_z x) - V_y (x + \varepsilon_z y, y - \varepsilon_z x) \varepsilon_z \\
V'_y &= V_y (x + \varepsilon_z y, y - \varepsilon_z x) + V_x (x + \varepsilon_z y, y - \varepsilon_z x) \varepsilon_z
\end{align*}
\]
This has the same form as [13] except that the angular momentum generator is now the sum of $L_z$ and the final matrix on the RHS above, which Shankar calls suggestively $S_z$, in anticipation of spin which at this stage he hasn’t considered. That is,

$$J_z = L_z + S_z \quad \quad (22)$$

$$= \begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} + \begin{bmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{bmatrix} \quad \quad (23)$$

The eigenvalues of the second matrix here are just $\pm \hbar$, so we haven’t yet encountered half-integral values of angular momentum.

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