SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES
FOR M=0

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Chapter 12, Exercise 12.5.10.
Post date: 6 Jun 2017

[If some equations are too small to read easily, use your browser’s magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

The total angular momentum operator $L^2$ can be written in spherical coordinates as

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$  \hspace{1cm} (1)

Since $[L^2, L_z] = 0$, we can find a basis consisting of simultaneous eigenfunctions of $L^2$ and $L_z$. Suppose we call these states $|\alpha \beta\rangle$, where $\alpha$ is the eigenvalue of $L^2$ and $\beta$ is the eigenvalue of $L_z$. In spherical coordinates, we know that

$$L_z = -i \hbar \frac{\partial}{\partial \phi}$$  \hspace{1cm} (2)

and that its eigenvalues are $m \hbar$ for integer values of $m$. Thus we can separate the $\theta$ and $\phi$ dependence in the eigenstates and write

$$\psi_{\alpha m}(\theta, \phi) = P^m_\alpha(\theta) e^{im\phi}$$  \hspace{1cm} (3)

We therefore have the eigenvalue equation

$$L^2 |\alpha m\rangle = \alpha |\alpha m\rangle$$  \hspace{1cm} (4)
$$L^2 \psi_{\alpha m}(\theta, \phi) = \alpha \psi_{\alpha m}(\theta, \phi)$$  \hspace{1cm} (5)

Combining (3) with (1) we have
SPHERICAL HARMONICS FROM POWER SERIES EXAMPLES FOR M=0

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi_{\alpha m}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi_{\alpha m}}{\partial \phi^2} = \alpha \psi_{\alpha m} \quad (6) \]

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P_{\alpha}^m}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} P_{\alpha}^m + \frac{\alpha}{\hbar^2} P_{\alpha}^m = 0 \quad (7) \]

We’d like to show that solutions of this equation require that (1)

\[ \alpha = \hbar^2 \ell (\ell + 1) \quad (8) \]

\[ |m| \leq \ell \quad (9) \]

for \( \ell = 0, 1, 2, \ldots \). In the problem given in Shankar, we tackle the less demanding case of \( m = 0 \) and demonstrate only the result for \( \alpha \). We begin by transforming (7) using the variable substitution:

\[ u \equiv \cos \theta \quad (10) \]

This gives us

\[ du = -\sin \theta \, d\theta \quad (11) \]

so that (7) becomes

\[ -\frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial u} \left( \sin \theta \frac{\partial P_{\alpha}^0}{\partial u} \right) + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0 \quad (12) \]

\[ \frac{\partial}{\partial u} \left( 1 - u^2 \right) \frac{\partial P_{\alpha}^0}{\partial u} + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0 \quad (13) \]

\[ (1 - u^2) \frac{\partial^2 P_{\alpha}^0}{\partial u^2} - 2u \frac{\partial P_{\alpha}^0}{\partial u} + \frac{\alpha}{\hbar^2} P_{\alpha}^0 = 0 \quad (14) \]

We can use a power series to solve this by defining

\[ P_{\alpha}^0 (u) = \sum_{n=0}^{\infty} C_n u^n \quad (15) \]

\[ \frac{\partial P_{\alpha}^0}{\partial u} = \sum_{n=0}^{\infty} C_n n u^{n-1} \quad (16) \]

\[ \frac{\partial^2 P_{\alpha}^0}{\partial u^2} = \sum_{n=0}^{\infty} C_n n(n-1) u^{n-2} \quad (17) \]

\[ = \sum_{n=0}^{\infty} C_{n+2} (n+2)(n+1) u^n \quad (18) \]
Plugging these into (14) and collecting terms, we get

\[
P^0_\alpha(u) = \sum_{n=0}^{\infty} \left[ C_{n+2} (n+2)(n+1) + C_n \left( -n(n-1) - 2n^2 + \frac{\alpha}{\hbar^2} \right) \right] u^n = 0
\]

(19)

If a power series equals zero, the coefficient of each power of \( u \) must be zero (power series theorem from math), so we get the recurrence relation

\[
C_{n+2} = \frac{n(n-1) + 2n - \frac{\alpha}{\hbar^2}}{(n+2)(n+1)} C_n
\]

(20)

\[
= \frac{n^2 + n - \frac{\alpha}{\hbar^2}}{n^2 + 3n + 2} C_n
\]

(21)

For large \( n \) we have

\[
C_{n+2} \to C_n \to \frac{n^2}{n^2} C_n = C_n
\]

(22)

Since \( u = \cos \theta \), \( u \in [-1, 1] \) and the series must converge for all these values. Although the power series \( \sum_{n=0}^{\infty} u^n \) converges if \( |u| < 1 \) (that’s the standard geometric series), it clearly diverges if \( u = 1 \). Thus we require the series to terminate, which imposes a condition on \( \alpha \):

\[
\alpha = \ell (\ell + 1) \hbar^2
\]

(23)

for some integer value \( \ell = 0, 1, 2, \ldots \). Since choosing a value for \( \ell \) can be done only once in any given series, and the recursion relation relates every second coefficient, this implies that either all even coefficients or all odd coefficients must be zero. Thus \( P^0_\alpha(u) \) is either a sum of even powers (making it an even function) or of odd powers (making it an odd function) only.

The first few values of \( P^0_\alpha(u) \) are found by choosing values for \( C_0 \) and \( C_1 \) and then generating all higher coefficients using (21). If we take

\[
C_0 = 1 \quad \text{(24)}
\]

\[
C_1 = 0 \quad \text{(25)}
\]

then if we choose \( \ell = 0 \) we get

\[
P^0_0 = 1 \quad \text{(26)}
\]

Taking
\[
C_0 = 0 \quad (27)
\]
\[
C_1 = 1 \quad (28)
\]

and \( \ell = 1 \) gives

\[
P_1^0 = u = \cos \theta \quad (29)
\]

Reverting to an even series and taking \( \ell = 2 \) we have from

\[
C_2 = -\frac{\alpha}{2\hbar^2} C_0 = -\frac{\ell (\ell + 1)}{2} (1) = -3 \quad (30)
\]
\[
P_2^0 = 1 - 3u^2 = 1 - 3\cos^2 \theta \quad (31)
\]

These values for \( P_\ell^0 \) agree with the spherical harmonics \( Y_\ell^0 \) apart from the constant scaling factors in each case. See Shankar’s equation 12.5.39 for comparison.

Pingbacks

Pingback: angular momentum and parity