NONDEGENERATE STATES IN 3-D: SPHERICALLY
SYMMETRIC SYSTEMS

In solving the Schrödinger equation for spherically symmetric potentials, we found that we could reduce the problem to the equation

\[
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \] \quad U_{El} = EU_{El} \tag{1}
\]

where \( U_{El}(r) \) is related to the radial function by

\[
R_{El}(r) = \frac{U_{El}(r)}{r} \tag{2}
\]

We can write [1] as an eigenvalue equation for the operator \( D_l \) in the form

\[
D_l(r) U_{El} = EU_{El} \tag{3}
\]

with

\[
D_l(r) \equiv -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \tag{4}
\]

We can show that, provided \( U_{El}(r) \to 0 \) as \( r \to 0 \), there are no degenerate eigenstates (that is, any state \( U_{El} \) that is an eigenstate with energy \( E \) is unique up to a scaling factor). The proof is similar to that in 1-d quantum mechanics, and goes by contradiction.

We suppose that there are two different functions \( U_1 \) and \( U_2 \) that satisfy [1] for the same energy \( E \) (and same angular momentum number \( l \)). We then have
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\[
\begin{bmatrix}
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}
\end{bmatrix} U_1 = EU_1
\] (5)

\[
\begin{bmatrix}
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}
\end{bmatrix} U_2 = EU_2
\] (6)

Multiply the first by \(U_2\) and the second by \(U_1\) and subtract to get

\[U_2 U''_1 - U_1 U''_2 = 0\] (7)

This expression is

\[U_2 U''_1 - U_1 U''_2 = \frac{d}{dr} (U_2 U'_1 - U_1 U'_2) = 0\] (8)

which we can integrate to get

\[U_2 U'_1 - U_1 U'_2 = C\] (9)

for some constant \(C\). This relation is valid for all \(r\), so we can choose \(r = 0\) where \(U_2 (0) = U_1 (0) = 0\), which shows that \(C = 0\). Therefore

\[\frac{U'_1}{U_1} = \frac{U'_2}{U_2}\] (10)

Integrating gives us

\[\ln U_1 = \ln U_2 + K\] (11)

for some other constant \(K\), so

\[U_1 = e^K U_2\] (12)

That is, any two eigenfunctions with the same eigenvalue \(E\) are multiples of each other, so represent the same state, which is nondegenerate.

Note that the derivation didn’t rely on the value of \(U\) anywhere except at \(r = 0\), so there is no requirement that, for example, \(U \to 0\) as \(r \to \infty\). Also, the derivation is valid whatever the sign of \(E\).