ISOTROPIC HARMONIC OSCILLATOR IN 3-D: USE OF SPHERICAL HARMONICS

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Chapter 12, Exercise 12.6.11.
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We’ve solved the 3-d isotropic harmonic oscillator before, so we’ve already solved most of Shankar’s exercise 12.6.11. We can quote the results here. The solution has the form

\[ \psi_{Elm} = \frac{U_{El}(r)}{r} Y_{lm}(\theta, \phi) \]  

The earlier solution uses notation from Griffiths’s book, but as the end result is the same, it’s not worth going through the derivation again using Shankar’s notation.

The potential is

\[ V(r) = \frac{1}{2} m\omega^2 r^2 \]  

The radial equation to be solved is

\[ \frac{d^2 u}{d\rho^2} = \left( -1 + \frac{l(l+1)}{\rho^2} + \rho_0^2 \rho^2 \right) u \]  

If we define

\[ \kappa^2 \equiv \frac{2\mu E}{\hbar^2} \]  
\[ \rho \equiv \kappa r \]  
\[ \rho_0 \equiv \frac{\mu \omega}{\hbar \kappa^2} = \frac{\hbar \omega}{2E} \]  

Taking the asymptotic behaviour of the radial function for small and large \( r \) into account leads us to a solution of form

\[ u(\rho) = \rho^{l+1} e^{-\rho_0 \rho^2 / 2} v(\rho) \]
Note that Griffiths’s $v$ is not the same as Shankar’s $v$, the latter of which is defined by Shankar’s equation 12.6.49. This gives a differential equation for Griffiths’s $v$

$$\frac{\rho}{d\rho^2} + 2 \left( l + 1 - \rho_0^2 \right) \frac{dv}{d\rho} + \rho (1 - \rho_0 (2l + 3)) v = 0$$

(8)

The function $v$ can be solved as a power series, giving

$$v(\rho) = \sum c_j \rho^j$$

(9)

Substituting into (8) leads to the recursion relation

$$c_{q+2} = \frac{\rho_0 (2q + 2l + 3) - 1}{(q + 2)(q + 2l + 3)} c_q$$

(10)

with $c_1 = 0$, so that $c_q = 0$ for all odd $q$. The requirement that the series terminates at some finite value of $j$ leads to the quantization condition on $E$:

$$E = \hbar \omega \left( q_{\text{max}} + l + \frac{3}{2} \right)$$

(11)

or, defining $n = q_{\text{max}} + l$,

$$E_n = \hbar \omega \left( n + \frac{3}{2} \right)$$

(12)

We worked out the degeneracies in the earlier post as well, so that the degeneracy of $E_n$ is

$$d(n) = \frac{1}{2} (n + 1)(n + 2)$$

(13)

To complete Shankar’s exercise, we need to work out the eigenfunctions for $n = 0$ and $n = 1$. For $n = 0$, $q_{\text{max}} = l = 0$, so only $c_0 \neq 0$ and we have

$$v(\rho) = c_0$$

(14)

$$u(\rho) = c_0 \rho e^{-\rho_0 \rho^2/2}$$

(15)

$$\psi_{000} = \frac{u}{r} Y_0^0$$

(16)

$$= c_0 \kappa e^{-\rho_0 \rho^2/2} Y_0^0$$

(17)

$$= c_0 \sqrt{\frac{2 \mu 3 \omega}{4 \pi \hbar}} e^{-\mu \omega \rho^2/2 \hbar}$$

(18)

where in the fourth line we used
\[ \kappa = \sqrt{\frac{2\mu E}{\hbar}} = \sqrt{\frac{2\mu \frac{3}{2} \hbar \omega}{\hbar}} = \sqrt{\frac{3\mu \omega}{\hbar}} \] (19)

\[ Y_0^0 = \frac{1}{\sqrt{4\pi}} \] (20)

Normalizing this requires that

\[ \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \psi_{000}^2 r^2 \sin \theta \, dr \, d\theta \, d\phi = c_0^2 \frac{6\mu \omega}{\hbar} \int_0^{\infty} e^{-\mu \omega r^2 / 2\hbar} \, dr \] (21)

\[ = 1 \] (22)

This is a standard Gaussian integral and can be done using software or tables so we get

\[ c_0 = \sqrt{\frac{6}{3}} \left( \frac{\mu \omega}{\pi \hbar} \right)^{1/4} \] (23)

This gives a wave function of

\[ \psi_{000} = \left( \frac{\mu \omega}{\pi \hbar} \right)^{3/4} e^{-\mu \omega r^2 / 2\hbar} \] (24)

which agrees with the earlier result.

For \( n = 1 \), the degeneracy is, from \( 13 \)

\[ d(1) = 3 \] (25)

The three possibilities are \( m = 0, \pm 1 \) which are reflected in the three spherical harmonics \( Y_1^{0, \pm 1} \). The radial function is the same in all cases, and is obtained from \( q_{\max} = 0, l = 1 \). From \( 7 \) this gives

\[ v(\rho) = c_0 \] (26)

\[ u(\rho) = c_0 \rho^2 e^{-\rho_0 \rho^2 / 2} \] (27)

\[ \psi_{11m} = \frac{u}{r} Y_1^m \] (28)

\[ = c_0 \kappa^2 r e^{-\rho_0 \rho^2 / 2} Y_1^m \] (29)

\[ = \frac{5\mu \omega}{\hbar} e^{-\mu \omega r^2 / 2\hbar} Y_1^m \] (30)

Again, we work out \( c_0 \) by imposing normalization. For example
\[ \psi_{111} = c_0 \frac{5 \mu \omega}{\hbar} r e^{-\mu \omega r^2 / 2\hbar} Y_1^1 \]  
\[ = -c_0 \frac{5 \mu \omega}{\hbar} r e^{-\mu \omega r^2 / 2\hbar} \sqrt{\frac{3}{8\pi}} \sin \theta e^{i \phi} \]  
(31)  
(32)

The normalization integral is

\[ \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_{111}^2 r^2 \sin \theta \, dr \, d\theta \, d\phi = c_0^2 \left( \frac{5 \mu \omega}{\hbar} \right)^2 \frac{3}{8\pi} \int_0^\pi \int_0^\infty e^{-\mu \omega r^2 / \hbar} r^4 \sin^3 \theta \, dr \, d\theta \]

\[ = c_0^2 \frac{75}{8} \sqrt{\frac{\pi \hbar}{\mu \omega}} = 1 \]  
(33)  
(34)  
(35)

I used Maple to do the integrals. This gives a wave function of

\[ \psi_{111} = -\sqrt{\frac{\mu \omega}{\hbar}} \left( \frac{\mu \omega}{\pi \hbar} \right)^{3/4} r e^{-\mu \omega r^2 / 2\hbar} \sin \theta e^{i \phi} \]  
(36)

We can work out the other two wave functions the same way (I used Maple, so I won’t go into the details):

\[ \psi_{11-1} = \sqrt{\frac{\mu \omega}{\hbar}} \left( \frac{\mu \omega}{\pi \hbar} \right)^{3/4} r e^{-\mu \omega r^2 / 2\hbar} \sin \theta e^{-i \phi} \]  
(37)

\[ \psi_{110} = \sqrt{\frac{2 \mu \omega}{\hbar}} \left( \frac{\mu \omega}{\pi \hbar} \right)^{3/4} r e^{-\mu \omega r^2 / 2\hbar} \cos \theta \]  
(38)

The \( \psi_{110} \) here is the same as \( \psi_{001} \) in our rectangular solution set. The other two are linear combinations of \( \psi_{100} \) and \( \psi_{010} \) from our rectangular set, which were (the suffixes in these 2 equations stand for \( x, y \) and \( z \), and not \( n, l \) and \( m \)):

\[ \psi_{100} = \sqrt{\frac{2 m \omega}{\hbar}} \left( \frac{m \omega}{\pi \hbar} \right)^{3/4} e^{-m \omega r^2 / 2\hbar} r \sin \theta \cos \phi \]  
(39)

\[ \psi_{010} = \sqrt{\frac{2 m \omega}{\hbar}} \left( \frac{m \omega}{\pi \hbar} \right)^{3/4} e^{-m \omega r^2 / 2\hbar} r \sin \theta \sin \phi \]  
(40)

We have
\[ \psi_{111} = \frac{1}{\sqrt{2}} (\psi_{100} + i\psi_{010}) \]  
\[ \psi_{11-1} = \frac{1}{\sqrt{2}} (\psi_{100} - i\psi_{010}) \]