PROJECTION OPERATORS FOR SPIN-1/2 + SPIN-1/2

We've seen projection operators in a formal mathematical sense, but in this post, we'll see a practical example of projection operators in spin space. We look at a system of two spin-$\frac{1}{2}$ particles, with spin operators $S_1$ and $S_2$ for each of the two particles. Now consider the operators

$$P_1 = \frac{3}{4}I + \frac{1}{\hbar^2}S_1 \cdot S_2$$

$$P_2 = \frac{1}{4}I - \frac{1}{\hbar^2}S_1 \cdot S_2$$

A projection operator projects an arbitrary vector onto a subspace of the vector space in which that vector resides. The two projection operators here project onto orthogonal subspaces, which means if we project some vector $V$ first with $P_1$ and then with $P_2$ (or vice versa), we'll end up with the zero vector. Also, if we project $V$ twice (or more) with the same projection operator, all projections after the first have no further effect. That is

$$P_i P_j = \delta_{ij} P_j$$

To show that this is true for the two projection operators above, we can make use of an identity derived earlier:

$$(\mathbf{A} \cdot \sigma)(\mathbf{B} \cdot \sigma) = (\mathbf{A} \cdot \mathbf{B})I + i(\mathbf{A} \times \mathbf{B}) \cdot \sigma$$

which is valid if $\mathbf{A}$ and $\mathbf{B}$ commute with $\sigma$.

Here $\mathbf{A}$ and $\mathbf{B}$ are vector operators that commute with the Pauli matrices $\sigma$.

First, we'll look at $P_1 P_2$: 

\[ \cdots \]
\[ P_1 P_2 = \left[ \frac{3}{4} I + \frac{1}{\hbar^2} S_1 \cdot S_2 \right] \left[ \frac{1}{4} I - \frac{1}{\hbar^2} S_1 \cdot S_2 \right] \]  

\[ = \left[ \frac{3}{4} I + \frac{1}{4} \sigma_1 \cdot \sigma_2 \right] \left[ \frac{1}{4} I - \frac{1}{4} \sigma_1 \cdot \sigma_2 \right] \]  

\[ = \frac{3}{16} I - \frac{2}{16} \sigma_1 \cdot \sigma_2 - \frac{1}{16} (\sigma_1 \cdot \sigma_2)^2 \]  

(5)  

(6)  

(7)  

We can write the last term as  

\[ (\sigma_1 \cdot \sigma_2)^2 = (\sigma_1 \cdot \sigma_2)(\sigma_1 \cdot \sigma_2) \]  

(8)  

We see that this has the same form as (4) with \( A = B = \sigma_1 \) and \( \sigma = \sigma_2 \). Since \( \sigma_1 \) and \( \sigma_2 \) refer to different spins, they commute, so the identity is valid. We get  

\[ (\sigma_1 \cdot \sigma_2)^2 = \sigma_1 \cdot \sigma_1 I + i (\sigma_1 \times \sigma_1) \cdot \sigma_2 \]  

(9)  

The first term is (using the fact that the square of each Pauli matrix is \( I \)):  

\[ \sigma_1 \cdot \sigma_1 I = (\sigma_{x_1}^2 + \sigma_{y_1}^2 + \sigma_{z_1}^2) I \]  

(10)  

\[ = 3 I^2 \]  

(11)  

\[ = 3 I \]  

(12)  

The cross product is just a shorthand way of writing the commutation relations. To see this, work out the \( x \) component, for example:  

\[ (\sigma_1 \times \sigma_1)_x = \sigma_{y_1} \sigma_{z_1} - \sigma_{z_1} \sigma_{y_1} = 2i \sigma_{x_1} \]  

(13)  

We can write this as  

\[ (\sigma_1 \times \sigma_1) = i \sigma_1 \]  

(14)  

Plugging this into (9) we have  

\[ (\sigma_1 \cdot \sigma_2)^2 = 3 I - 2 \sigma_1 \cdot \sigma_2 \]  

(15)  

This gives, from (7)  

\[ P_1 P_2 = \frac{3}{16} I - \frac{2}{16} \sigma_1 \cdot \sigma_2 - \frac{3}{16} I + \frac{2}{16} \sigma_1 \cdot \sigma_2 = 0 \]  

(16)  

A similar calculation shows that  

\[ P_2 P_1 = 0 \]  

(17)  

We can also calculate
PROJECTION OPERATORS FOR SPIN-1/2 + SPIN-1/2

\[ P_1 P_1 = \left[ \frac{3}{4} I + \frac{1}{h^2} S_1 \cdot S_2 \right] \left[ \frac{3}{4} I + \frac{1}{h^2} S_1 \cdot S_2 \right] \]  
\[ = \left[ \frac{3}{4} I + \frac{1}{4} \sigma_1 \cdot \sigma_2 \right] \left[ \frac{3}{4} I + \frac{1}{4} \sigma_1 \cdot \sigma_2 \right] \]  
\[ = \frac{9}{16} I + \frac{6}{16} \sigma_1 \cdot \sigma_2 + \frac{1}{16} (\sigma_1 \cdot \sigma_2)^2 \]  
\[ = \frac{12}{16} I + \frac{4}{16} \sigma_1 \cdot \sigma_2 \]  
\[ = \frac{3}{4} I + \frac{1}{4} \sigma_1 \cdot \sigma_2 \]  
\[ = P_1 \]  

(18)

(19)

(20)

(21)

(22)

(23)

A similar calculation shows that

\[ P_2 P_2 = P_2 \]  

(24)

To find the subspace to which each projection operator projects, we can use the explicit matrix forms in the product basis for the projections. We have

\[ P_1 = \frac{3}{4} I + \frac{1}{h^2} S_1 \cdot S_2 \]  
\[ = \frac{3}{4} I + \frac{1}{h^2} \left( \frac{1}{2} S_{1+} S_{2-} + \frac{1}{2} S_{1-} S_{2+} + S_{1z} S_{2z} \right) \]  
\[ = \frac{3}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \]  
\[ \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

(25)

(26)

(27)

(28)

(29)

Similarly
\[ P_2 = \frac{1}{4} I - \frac{1}{\hbar^2} S_1 \cdot S_2 \]
\[ = \frac{3}{4} I - \frac{1}{\hbar^2} \left( \frac{1}{2} S_{1+} S_{2-} + \frac{1}{2} S_{1-} S_{2+} + S_{1z} S_{2z} \right) \]
\[ = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & \frac{1}{2} (b+c) & 0 \\ 0 & \frac{1}{2} (b+c) & 0 \\ 0 & 0 & d \end{bmatrix} \]
\[ = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} (b+c) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

Thus \( P_1 \) projects \( V \) into the subspace spanned by the basis vectors of the 3-dimensional spin-1 subspace.

For \( P_2 \) we have
\[ P_2 V = P_2 \begin{bmatrix} a & b & c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \] (38)

\[ = \frac{1}{2} \begin{bmatrix} 0 \\ -b + c \\ b - c \end{bmatrix} \] (39)

\[ = \frac{1}{\sqrt{2}} (b - c) \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \] (40)

Thus \( P_2 \) projects onto the 1-dimensional spin-0 subspace.

In the total-\( j \) basis

\[ S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2 \] (41)

\[ S_1 \cdot S_2 = \frac{1}{2} (S^2 - S_1^2 - S_2^2) \] (42)

In both the spin-1 and spin-0 states, the eigenvalues of \( S_1^2 \) and \( S_2^2 \) are equal to \( s_1 (s_1 + 1) \hbar^2 = \frac{3}{4} \). For spin-1, \( s = 1 \) and for the three basis states with \( m = \pm 1, 0 \), we have, since all operators are diagonal in this space:

\[ (S_1 \cdot S_2) |s = 1, m = \pm 1, 0\rangle = \frac{1}{2} (S^2 - S_1^2 - S_2^2) |s = 1, m = \pm 1, 0\rangle \] (43)

\[ = \frac{\hbar^2}{2} \left( s (s + 1) - \frac{3}{2} \right) I |s = 1, m = \pm 1, 0\rangle \] (44)

\[ = \frac{\hbar^2}{4} |s = 1, m = \pm 1, 0\rangle \] (45)

For the spin-0 state, there is only one basis state with \( m = 0 \), so
\[ (S_1 \cdot S_2) |s = 0, m = 0\rangle = \frac{1}{2} (S_1^2 - S_2^2 - S_2^2) |s = 0, m = 0\rangle \]
\[ = \frac{\hbar^2}{2} \left( s(s + 1) - \frac{3}{2} \right) |s = 0, m = 0\rangle \]
\[ = -\frac{3\hbar^2}{4} |s = 0, m = 0\rangle \]

Therefore, on any spin-1 state, we have

\[ \mathbb{P}_1 |s = 1, m = \pm 1, 0\rangle = \left( \frac{3}{4} I + \frac{1}{\hbar^2} S_1 \cdot S_2 \right) |s = 1, m = \pm 1, 0\rangle \]
\[ = \left( \frac{3}{4} + \frac{1}{4} \right) I |s = 1, m = \pm 1, 0\rangle \]
\[ = |s = 1, m = \pm 1, 0\rangle \]
\[ \mathbb{P}_2 |s = 1, m = \pm 1, 0\rangle = \left( \frac{1}{4} I - \frac{1}{\hbar^2} S_1 \cdot S_2 \right) |s = 1, m = \pm 1, 0\rangle \]
\[ = \left( \frac{1}{4} - \frac{1}{4} \right) I |s = 1, m = \pm 1, 0\rangle \]
\[ = 0 \]

On the spin-0 state

\[ \mathbb{P}_1 |s = 0, m = 0\rangle = \left( \frac{3}{4} I + \frac{1}{\hbar^2} S_1 \cdot S_2 \right) |s = 0, m = 0\rangle \]
\[ = \left( \frac{3}{4} - \frac{3}{4} \right) I |s = 0, m = 0\rangle \]
\[ = 0 \]
\[ \mathbb{P}_2 |s = 0, m = 0\rangle = \left( \frac{1}{4} I - \frac{1}{\hbar^2} S_1 \cdot S_2 \right) |s = 0, m = 0\rangle \]
\[ = \left( \frac{1}{4} + \frac{3}{4} \right) I |s = 0, m = 0\rangle \]
\[ = |s = 0, m = 0\rangle \]

Since the four kets \(|s = 1, m = \pm 1, 0\rangle\) and \(|s = 0, m = 0\rangle\) form a basis in the total-j space, any state can be written as a linear combination of them, and thus the projection operator \(\mathbb{P}_1\) projects an arbitrary vector onto the \(|s = 1, m = \pm 1, 0\rangle\) subspace and \(\mathbb{P}_2\) onto the \(|s = 0, m = 0\rangle\) subspace.
PINGBACKS

Pingback: Projection operators for general L + spin-1/2