SPHERICAL TENSOR OPERATORS; COMMUTATORS

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Post date: 20 Oct 2017

A spherical tensor operator is defined to be an object $T^q_k$ with integer indices $q$ and $k$. The rank of the tensor is $k$, and the other index $q$ ranges in integer steps from $-k$ to $+k$, giving $2k + 1$ components. Its definition includes a requirement that it transform under a rotation according to

$$U[R] T^q_k U^\dagger[R] = \sum_{q'} D^{(k)}_{qq'} T^{q'}_k$$  \hspace{1cm} (1)

where $D^{(k)}$ is the $k$-th block in the block diagonal matrix formed from the angular momentum operators $J$. For a rotation through an angle $\theta$ about an axis $\hat{\theta}$, we have

$$D^{(k)}[R(\theta)] = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-i \theta}{\hbar} \right)^n \left( \hat{\theta} \cdot J^{(k)} \right)^n$$  \hspace{1cm} (2)

where $J^{(k)}$ is the angular momentum vector obtained from the $k$-th block in each of $J_x$, $J_y$ and $J_z$ (see Shankar section 12.5 for details).

The series can be written in closed form for some small values of $k$, but we won’t need these forms here.

For a set of angular momentum kets $|kq\rangle$ (Shankar changes the notation here, in that $|kq\rangle$ refers to a state with total angular momentum number $k$ and $z$ component $q$, rather than the more familiar $|jm\rangle$), the matrix elements of $D^{(k)}$ are

$$D^{(k)}_{qq'} = \langle kq' | U[R] | kq \rangle$$  \hspace{1cm} (3)

Note that

$$\langle k'q' | U[R] | kq \rangle = D^{(k)}_{qq'} \delta_{k'k}$$  \hspace{1cm} (4)

This follows because a rotation cannot change the total angular momentum of a state, so $U[R] |kq\rangle$ will always result in a state whose total angular momentum number is also $k$. From this fact, we can write the rotation of an angular momentum ket as

$1$
Comparing this result with 1, we see that a passive transformation of the tensor operator $T^q_k$ works in the same way as a rotation of an angular momentum eigenstate $|kq\rangle$.

We can use 1 to work out the commutators of $T^q_k$ with the components of the angular momentum operator $J$. We use the fact that angular momentum is the generator of rotations and consider an infinitesimal rotation $\delta \theta \theta \theta$ about, say, the $x$ axis. In this case, working to first order in $\delta \theta$:

$$U[R] = I - \frac{i\delta \theta J_x}{\hbar}$$
$$U^\dagger[R] = I + \frac{i\delta \theta J_x}{\hbar}$$

$$U[R] T^q_k U^\dagger[R] = \left( I - \frac{i\delta \theta J_x}{\hbar} \right) T^q_k \left( I + \frac{i\delta \theta J_x}{\hbar} \right)$$

On the RHS of 11 we can use 3 to first order in $\delta \theta$:

$$D_{q'q}^{(k)} T^q_k = \left( kq' \right) \left[ J_x, T^q_k \right]$$

Combining the last two results, we have

$$[J_x, T^q_k] = \sum_{q'} \left( kq' \left| J_x \right| kq \right) T^q_k$$

We could do the same analysis for the $y$ and $z$ components, and we’d get the same result, so we have
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\[
\begin{align*}
[J_y, T^q_k] &= \sum_{q'} \langle kq' | J_y | kq \rangle T^{q'}_k \\
[J_z, T^q_k] &= \sum_{q'} \langle kq' | J_z | kq \rangle T^{q'}_k
\end{align*}
\] (16) (17)

We can simplify the last equation, since the ket \( |kq\rangle \) is an eigenket of \( J_z \) with eigenvalue \( q\hbar \). We therefore have

\[
\sum_{q'} \langle kq' | J_z | kq \rangle T^{q'}_k = \sum_{q'} \langle kq' | kq \rangle \hbar q T^q_k
\] (18)

\[
= \hbar q T^q_k
\] (19)

To deal with the other two components, we can combine the results in [15] and [16] and use the raising and lowering operators.

\[ J_\pm = J_x \pm iJ_y \] (20)

\[ J_\pm |kq\rangle = \hbar \sqrt{(k \mp q)(k \pm q + 1)} |k, q \pm 1\rangle \] (21)

We have

\[
[J_\pm, T^q_k] = \sum_{q'} \langle kq' | J_\pm | kq \rangle T^{q'}_k
\] (22)

\[
= \hbar \sqrt{(k \mp q)(k \pm q + 1)} \sum_{q'} \langle kq' | k, q \pm 1 \rangle T^{q'}_k
\] (23)

\[
= \hbar \sqrt{(k \mp q)(k \pm q + 1)} T^{q \pm 1}_k
\] (24)

where we’ve again used the orthogonality of the eigenkets to get the last line.

Example. Suppose we construct a spherical tensor out of the components of a vector operator \( V \) so that we have a rank 1 tensor given by

\[
T^{\pm 1}_1 = \pm \frac{V_x \pm iV_y}{\sqrt{2}}
\] (25)

\[
T^0_1 = V_z
\] (26)

Vector operators obey the commutation rules

\[
[V_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} V_k
\] (27)

Applying this gives us, for example
\[
[T_1^1, J_x] = -\frac{1}{\sqrt{2}} (\{V_x, J_x\} + i \{V_y, J_x\})
\]
\[
= -\frac{1}{\sqrt{2}} (0 + \hbar V_z)
\]
\[
= -\hbar \frac{V_z}{\sqrt{2}}
\]
\[
[T_1^1, J_y] = -\frac{1}{\sqrt{2}} (\{V_x, J_y\} + i \{V_y, J_y\})
\]
\[
= -\frac{1}{\sqrt{2}} (i \hbar V_z + 0)
\]
\[
= -i \hbar \frac{V_z}{\sqrt{2}}
\]

Combining these results, we have

\[
[T_1^1, J_+ ] = [T_1^1, J_x] + i [T_1^1, J_y]
\]
\[
= -\hbar \frac{V_z}{\sqrt{2}} + \hbar \frac{V_z}{\sqrt{2}}
\]
\[
= 0
\]

This agrees with 24 with \(k = q = 1\).
We also have

\[
[T_1^1, J_- ] = [T_1^1, J_x] - i [T_1^1, J_y]
\]
\[
= -\hbar \frac{V_z}{\sqrt{2}} - \hbar \frac{V_z}{\sqrt{2}}
\]
\[
= -\sqrt{2} \hbar V_z
\]
\[
= -\sqrt{2} \hbar T_1^0
\]

This also agrees with 24 with \(k = q = 1\) (since \([T_1^1, J_- ] = -[J_-, T_1^1]\)).

We can do similar calculations to find that

\[
[T_1^{-1}, J_+] = -\sqrt{2} \hbar T_1^0
\]
\[
[T_1^{-1}, J_- ] = 0
\]

Finally, we have
\[ [T_1^1, J_z] = -\frac{1}{\sqrt{2}} ([V_x, J_z] + i [V_y, J_z]) \]  
\[ = -\frac{1}{\sqrt{2}} (-i\hbar V_y - \hbar V_x) \]  
\[ = \frac{\hbar}{\sqrt{2}} (V_x + iV_y) \]  
\[ = -\hbar T_1^1 \]  
\[ [J_z, T_1^1] = \hbar T_1^1 \]  

which is again consistent with [19] with \( q = 1 \). Similar calculations can be done to verify the other commutation relations.