Here we’ll apply the variational principle again to the harmonic oscillator, this time with potential

$$\psi(x) = \begin{cases} 
(x-a)^2 (x+a)^2 & x \leq |a| \\
0 & |x| > a 
\end{cases}$$

where $a$ is the parameter to be varied, and we can see that it controls the width of the trial wave function as well as its height. We first find the normalization constant

$$N \equiv \langle \psi | \psi \rangle = \int_{-a}^{a} (x-a)^4 (x+a)^4 \, dx$$

where I used Maple to do and simplify the integral. If you want to do it by hand, it’s probably easiest to use the substitution $u = x - a$ before multiplying out the factors in the integrand.

The energy estimate is then obtained by minimizing

$$E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

where the Hamiltonian contains the harmonic oscillator potential:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2$$

To calculate $\langle \psi | H | \psi \rangle$ requires integrating a sixth-degree polynomial which is straightforward but very tedious to do by hand if you like (which is probably why the exercise is marked as 'optional' in Shankar), but again I used Maple to get
\[
\frac{d^2 \psi}{dx^2} = 12x^2 - 4a^2 
\]
\[\langle \psi \mid H \mid \psi \rangle = \int_{-a}^{a} (x-a)^2 (x+a)^2 \left[ -\frac{\hbar^2}{2m} \right] 12x^2 - 4a^2 + 2m\omega^2 x^2 (x-a)^2 (x+a)^2 \, dx \]
\[
= \frac{128}{3465} \left( a^{11}m\omega^2 + 33\frac{\hbar^2 a^7}{m} \right) 
\]

The expression to minimize is therefore

\[
E = \frac{128}{3465} \left( a^{11}m\omega^2 + 33\frac{\hbar^2 a^7}{m} \right) \times \frac{315}{256a^8} 
\]
\[
= \frac{1}{22} \left( a^2m\omega^2 + \frac{33\hbar^2}{m}a^{-2} \right) 
\]

Taking the derivative, we need to solve

\[
\frac{dE}{da} = \frac{1}{11} \left( am\omega^2 - 33\frac{\hbar^2}{m}a^{-3} \right) = 0 
\]

This gives an optimum value for \( a \):

\[
a_0 = 33^{1/4} \sqrt{\frac{\hbar}{m\omega}} 
\]

Substituting into \( \boxed{11} \) we get the estimate of the ground state energy

\[
E_0 = \frac{\sqrt{33}}{11} \hbar\omega \simeq 0.522\hbar\omega 
\]

The exact ground state energy for the harmonic oscillator is \( \frac{1}{2}\hbar\omega \) so this estimate is reasonably good.

This answer agrees with the back-of-the-book answer in Shankar, since

\[
\frac{\sqrt{33}}{11} = \frac{1}{2} \frac{\sqrt{4 \times 33}}{11} = \frac{1}{2} \frac{\sqrt{12 \times 11}}{11} = \frac{1}{2} \sqrt{\frac{12}{11}} 
\]