GENERATORS OF THE Lorentz GROUP - MOMENTUM COMMITATORS

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References: Mark Srednicki, Quantum Field Theory, (Cambridge University Press, 2007) - Chapter 2, Problems 2.5 - 2.6.

The four-momentum $P_\mu$ transforms under a Lorentz transformation according to Srednicki’s equation 2.15:

$$U^{-1}(\Lambda) P_\mu U(\Lambda) = \Lambda_\mu^\nu P_\nu \quad (1)$$

Under an infinitesimal transformation, the unitary operators have the form

$$U(I + \delta \omega) = I + \frac{i}{2\hbar} \delta \omega_{\mu\nu} M^{\mu\nu} \quad (2)$$

By using the same technique that we used to find the commutators of $M^{\mu\nu}$ we can find the commutators of $P_\mu$. The LHS of (1) is, to first order in $\delta \omega$:

$$U^{-1}(\Lambda) P_\mu U(\Lambda) = \left( I - \frac{i}{2\hbar} \delta \omega_{\rho\sigma} M^{\rho\sigma} \right) P_\mu \left( I + \frac{i}{2\hbar} \delta \omega_{\rho\sigma} M^{\rho\sigma} \right) \quad (3)$$

$$= P_\mu + \frac{i}{2\hbar} \delta \omega_{\rho\sigma} [P_\mu, M^{\rho\sigma}] \quad (4)$$

The RHS of (1) is

$$\Lambda_\mu^\nu P_\nu = (\delta_\mu^\nu + \delta \omega_{\nu}^\nu) P_\nu \quad (5)$$

$$= P_\mu + g^{\mu\rho} \delta \omega_{\rho\sigma} P_\sigma \quad (6)$$

Comparing with (4), we can cancel the $P_\mu$ term on both sides. The remaining term on each side has the form of $\delta \omega_{\rho\sigma}$ multiplied by a coefficient. The term $\delta \omega_{\rho\sigma} [P_\mu, M^{\rho\sigma}]$ in (4) is symmetric under the interchange $\rho \leftrightarrow \sigma$ (since both $\delta \omega_{\rho\sigma}$ and $M^{\rho\sigma}$ are antisymmetric), so swapping $\rho \leftrightarrow \sigma$ leaves this term unchanged. Swapping $\rho \leftrightarrow \sigma$ in (6) does give us something different however. If we swap $\rho \leftrightarrow \sigma$ in both equations and then add the results we get
\[
\frac{i}{\hbar} \delta \omega_{\rho \sigma} [P^\mu, M^{\rho \sigma}] = g^{\mu \rho} \delta \omega_{\rho \sigma} P^\sigma + g^{\mu \sigma} \delta \omega_{\rho \sigma} P^\rho
\]
(7)

\[
= g^{\mu \rho} \delta \omega_{\rho \sigma} P^\sigma - g^{\mu \sigma} \delta \omega_{\rho \sigma} P^\rho
\]
(8)

\[
= \delta \omega_{\rho \sigma} (g^{\mu \rho} P^\sigma - g^{\mu \sigma} P^\rho)
\]
(9)

Since \( \delta \omega_{\rho \sigma} \) is arbitrary, its coefficients must be equal on each side of the equation. Multiplying through by \(-i\hbar\) we get Srednicki’s equation 2.18:

\[
[P^\mu, M^{\rho \sigma}] = i\hbar (g^{\mu \sigma} P^\rho - g^{\mu \rho} P^\sigma)
\]
(10)

Using this formula, we can work out the commutators of the angular momentum and boost operators with \( P^\mu \). We recall that

\[
J_i = \frac{1}{2} \varepsilon_{ijk} M^{jk}
\]
(11)

\[
K_i = M^{i0}
\]
(12)

The energy \( H \) is \( cP^0 \) so using (10) we have, for example

\[
[J_1, H] = \frac{c}{2} [M^{23} - M^{32}, P^0]
\]
(13)

\[
= c [M^{23}, P^0]
\]
(14)

\[
= -c [P^0, M^{23}]
\]
(15)

\[
= 0
\]
(16)

where the last line follows because we set \( \mu = 0, \rho = 2 \) and \( \sigma = 3 \) in (10) which causes all the \( g \) terms to be zero, since \( g^{\mu \nu} \) is diagonal. The same result follows for \([J_2, H] = [J_3, H] = 0\).

Now consider

\[
[J_1, P_2] = [M^{23}, P^2]
\]
(17)

\[
= -i\hbar (g^{23} P^2 - g^{22} P^3)
\]
(18)

\[
= i\hbar P^3
\]
(19)

We can cycle the indexes to get the other commutation relations with the general result (for spatial indexes 1,2,3, \( P^i = P_i \)):

\[
[J_i, P_j] = i\hbar \varepsilon_{ijk} P_k
\]
(20)

For the boosts we have
\[ [K_1, H] = c [M^{10}, P^0] \]
\[ = -i\hbar (g^{00}P^1 - g^{01}P^0) \]
\[ = i\hbar P^1 \]

since \( g^{00} = -1 \). Cyclic permutation gives the general result

\[ [K_i, H] = i\hbar P_i \] (24)

Finally we consider

\[ [K_1, P_2] = [M^{10}, P^2] \]
\[ = -i\hbar (g^{20}P^1 - g^{21}P^0) \]
\[ = 0 \] (27)

and

\[ [K_1, P_1] = [M^{10}, P^1] \]
\[ = -i\hbar (g^{10}P^1 - g^{11}P^0) \]
\[ = i\hbar P^0 \]
\[ = i\hbar \frac{H}{c} \] (31)

The general result is

\[ [K_i, P_j] = \frac{i\hbar}{c} \delta_{ij} H \] (32)