In the Heisenberg picture, the time dependence of a quantum system resides in the operators, rather than in the wave functions or states. In nonrelativistic theory, the time evolution of a state is given by

\[ \phi(x, t) = e^{iHt/\hbar} \phi(x, 0) e^{-iHt/\hbar} \]  

where \( H \) is the Hamiltonian. The relativistic generalization is

\[ \phi(x) = e^{-iPx/\hbar} \phi(0) e^{iPx/\hbar} \]  

where \( P \) and \( x \) are the momentum and spacetime four-vectors. By defining the spacetime translation operator as

\[ T(a) \equiv \exp(-iP \mu a_{\mu}/\hbar) \]  

where \( a_{\mu} \) is a spacetime four-vector we can write as

\[ \phi(a) = T(a) \phi(0) T^{-1}(a) \]  

or, if we start at location \( x - a \), the translation \( T(a) \) moves us to location \( x \). We can write the inverse of this transformation as

\[ \phi(x - a) = T^{-1}(a) \phi(x) T(a) \]  

Srednicki then draws an analogy between this general spacetime transformation and the Lorentz transformation to write his equation 2.26, where we have, for a Lorentz transformation \( \Lambda \)

\[ U^{-1}(\Lambda) \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x) \]  

Another way of writing this to get a forward transformation is

\[ \phi(x) = U(\Lambda) \phi(\Lambda^{-1}x) U^{-1}(\Lambda) \]  

We’ve seen that for an infinitesimal transformation
\[ U (I + \delta \omega) = I + \frac{i}{\hbar} \delta \omega_{\mu\nu} M^{\mu\nu} \]  

where \( M^{\mu\nu} = - M^{\nu\mu} \) are the generators of the Lorentz group. We’ve also seen that the commutators are given by

\[ [M^{\mu\nu}, M^{\rho\sigma}] = i\hbar ((g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma}) - (g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho})) \]  

Another way of deriving this begins with \( \delta \) for an infinitesimal transformation. We have

\[ \left( I - \frac{i}{\hbar} \delta \omega_{\mu\nu} M^{\mu\nu} \right) \phi(x) \left( I + \frac{i}{\hbar} \delta \omega_{\mu\nu} M^{\mu\nu} \right) = \phi ( (I - \delta \omega_{\mu\nu} ) x^{\nu} ) \]  

The LHS can be expanded in the same way we used earlier to get, to first order in \( \delta \omega_{\mu\nu} \)

\[ LHS = \phi (x) + \frac{i}{\hbar} \delta \omega_{\mu\nu} (\phi(x) M^{\mu\nu} - M^{\mu\nu} \phi(x)) \]  

The RHS of 10 can be expanded in a Taylor series to the same order:

\[ \phi ( (I - \delta \omega_{\mu\nu} ) x^{\nu} ) = \phi (x) - \delta \omega_{\mu\nu} x^{\nu} \partial^{\mu} \phi(x) \]  

We can cancel the \( \phi(x) \) from both sides to get

\[ \frac{i}{\hbar} \delta \omega_{\mu\nu} (\phi(x) M^{\mu\nu} - M^{\mu\nu} \phi(x)) = - \delta \omega_{\mu\nu} x^{\nu} \partial^{\mu} \phi(x) \]  

Because both \( \delta \omega_{\mu\nu} \) and \( M^{\mu\nu} \) are antisymmetric, we can swap \( \mu \leftrightarrow \nu \) on the LHS of this equation, leaving it unchanged. However, the RHS does change under this swap, so if we add the original equation to its swapped counterpart, we get

\[ \frac{i}{\hbar} \delta \omega_{\mu\nu} (\phi(x) M^{\mu\nu} - M^{\mu\nu} \phi(x)) = - \delta \omega_{\mu\nu} x^{\nu} \partial^{\mu} \phi(x) + \delta \omega_{\mu\nu} x^{\mu} \partial^{\nu} \phi(x) \]  

Multiplying through by \( \frac{\hbar}{i} \) and equating coefficients of \( \delta \omega_{\mu\nu} \) we get

\[ \phi(x) M^{\mu\nu} - M^{\mu\nu} \phi(x) = \frac{\hbar}{i} (x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}) \phi(x) \]  

or

\[ U (I + \delta \omega) = I + \frac{i}{\hbar} \delta \omega_{\mu\nu} M^{\mu\nu} \]
GENERATORS OF THE LORENTZ GROUP - ALTERNATIVE DERIVATION OF THE COMMUTATORS

\[ [\phi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x) = \mathcal{L}^{\mu\nu} \phi(x) \]  
where

\[ \mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu) \]  

We can now work out the following.

\[ [[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = (\mathcal{L}^{\mu\nu} \phi(x)) M^{\rho\sigma} - M^{\rho\sigma} \mathcal{L}^{\mu\nu} \phi(x) \]  
\[ = \mathcal{L}^{\mu\nu} [\phi(x), M^{\rho\sigma}] \]  
\[ = \mathcal{L}^{\mu\nu} \mathcal{L}^{\rho\sigma} \phi(x) \]  

We’re justified in taking \( \mathcal{L}^{\mu\nu} \) outside the commutator in the second line, since \( \mathcal{L}^{\mu\nu} \) operates only on functions of \( x \), and \( M^{\rho\sigma} \) does not depend on \( x \).

Srednicki then asks us to prove the Jacobi identity for the commutators of three operators, which is

\[ [[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \]  

This can be proved by brute force by just writing out all the commutators in full and then finding that the terms cancel in pairs. I won’t bother with this as it gets quite tedious. Just note that, for example

\[ [[A, B], C] = [A, B] C - C [A, B] \]  
\[ = ABC - BAC - CAB + CBA \]  

and so on for the other two.

We can now use 22 and 23 to derive the following.

\[ [\phi, [M^{\mu\nu}, M^{\rho\sigma}]] = -[[M^{\mu\nu}, M^{\rho\sigma}], \phi] \]  
\[ = [[M^{\rho\sigma}, \phi], M^{\mu\nu}] + [[\phi, M^{\mu\nu}], M^{\rho\sigma}] \]  
\[ = -\mathcal{L}^{\rho\sigma} \mathcal{L}^{\mu\nu} \phi(x) + \mathcal{L}^{\mu\nu} \mathcal{L}^{\rho\sigma} \phi(x) \]  
\[ = (\mathcal{L}^{\mu\nu} \mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma} \mathcal{L}^{\mu\nu}) \phi(x) \]  

To simplify this, we need to work out the \( \mathcal{L} \) operators as they act on \( \phi(x) \), using its definition 19. To do this, we first note that, since the \( x^\nu \) are independent variables

\[ \partial^\mu x^\nu = g^{\mu\nu} \]  

We can do the tedious derivatives, using the product rule where required. For the first term in 29 we have
\[ L^{\mu \nu} L^{\rho \sigma} \phi (x) = -\hbar^2 [x^{\mu} (g^{\nu \rho} \partial^{\sigma} \phi + x^{\nu} \partial^{\rho} \phi - g^{\nu \sigma} \partial^{\rho} \phi - x^{\rho} \partial^{\nu} \phi)] \quad (31) \]

\[ + \hbar^2 [x^{\nu} (g^{\mu \rho} \partial^{\sigma} \phi + x^{\mu} \partial^{\rho} \phi - g^{\mu \sigma} \partial^{\rho} \phi - x^{\rho} \partial^{\mu} \phi)] \quad (32) \]

For the second term, we swap \( \mu \leftrightarrow \rho \) and \( \nu \leftrightarrow \sigma \):

\[ -L^{\sigma \rho} L^{\mu \nu} \phi (x) = \hbar^2 [x^{\rho} (g^{\sigma \nu} \partial^{\mu} \phi + x^{\sigma} \partial^{\nu} \phi - g^{\sigma \mu} \partial^{\nu} \phi - x^{\nu} \partial^{\sigma} \phi)] \quad (33) \]

\[ -\hbar^2 [x^{\sigma} (g^{\rho \mu} \partial^{\nu} \phi + x^{\rho} \partial^{\mu} \phi - g^{\rho \nu} \partial^{\mu} \phi - x^{\mu} \partial^{\rho} \phi)] \quad (34) \]

Adding these two terms, we see that all the second derivative terms cancel, and since \( g^{\mu \nu} = g^{\nu \mu} \), we can group terms to get

\[ (L^{\mu \nu} L^{\rho \sigma} - L^{\rho \sigma} L^{\mu \nu}) \phi (x) = i\hbar [g^{\nu \rho} (x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}) + g^{\nu \sigma} (x^{\mu} \partial^{\sigma} - x^{\sigma} \partial^{\mu})] \quad (35) \]

\[ + i\hbar [g^{\mu \rho} (x^{\nu} \partial^{\sigma} - x^{\sigma} \partial^{\nu}) + g^{\mu \sigma} (x^{\nu} \partial^{\rho} - x^{\rho} \partial^{\nu})] \quad (36) \]

Comparing this with [18] we find

\[ [\phi, [M^{\mu \nu}, M^{\rho \sigma}]] = i\hbar (g^{\nu \rho} [\phi, M^{\sigma \mu}] + g^{\nu \sigma} [\phi, M^{\rho \mu}] + g^{\mu \rho} [\phi, M^{\nu \sigma}] + g^{\mu \sigma} [\phi, M^{\rho \nu}]) \quad (37) \]

\[ = i\hbar \left[ \phi, (g^{\mu \rho} M^{\nu \sigma} - g^{\nu \rho} M^{\mu \sigma}) - (g^{\mu \sigma} M^{\nu \rho} - g^{\nu \sigma} M^{\mu \rho}) \right] \quad (38) \]

where we’ve used the antisymmetry of \( M^{\sigma \mu} \) and \( M^{\rho \nu} \) to get the last line. We thus find that

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i\hbar \left( (g^{\mu \rho} M^{\nu \sigma} - g^{\nu \rho} M^{\mu \sigma}) - (g^{\mu \sigma} M^{\nu \rho} - g^{\nu \sigma} M^{\mu \rho}) \right) + A \quad (39) \]

where \([\phi, A] = 0\), which agrees with [9] up to the possible factor \( A \).