PARTICLE MOVING IN A CIRCULAR ORBIT

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.2.

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Consider a particle moving in a circular orbit of radius r at a constant speed $v = |\boldsymbol{v}|$ and constant magnitude of acceleration $a = |\boldsymbol{a}|$. We now define a unit vector along the direction of the velocity, so that

$$\boldsymbol{n} \equiv \frac{\boldsymbol{v}}{v} \tag{1}$$

and further define s to be the distance along the circle that the particle moves, so that

$$v = \frac{ds}{dt} = r\omega \tag{2}$$

where ω is the angular speed.

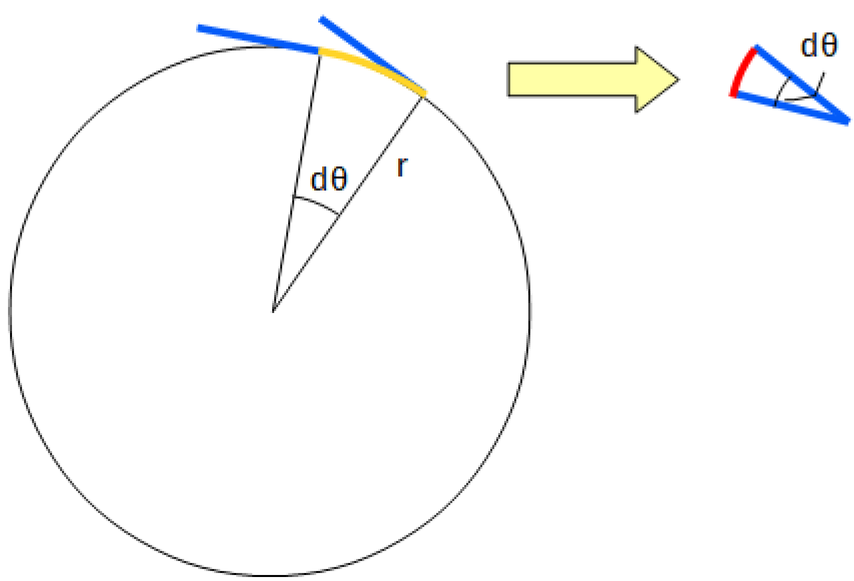

FIGURE 1. Particle in a circular orbit.

Ex. 1.2(a). We want to find $d\boldsymbol{n}/ds$. We can write this as

$$\frac{d\boldsymbol{n}}{ds}=\frac{d\boldsymbol{n}}{dt}\frac{dt}{ds}=\frac{1}{v}\frac{d\boldsymbol{n}}{dt} \tag{3}$$

Referring to Fig. 1, we've drawn in yellow the distance around the circle that the particle moves in a time dt. In that time, it subtends an angle of $d\theta$, so the distance along the arc is $r\ d\theta$. The velocity is always tangent to the circle, so we've drawn in blue the vector $\boldsymbol{n}$ at the start and end of the particle's journey through the yellow arc. Since the velocity is perpendicular to the radius, we can move these two $\boldsymbol{n}$ vectors so that they start at the same point (on the RHS of the figure) and thus the angle between them is also $d\theta$. The length of the red arc is therefore just $d\theta$, since $\boldsymbol{n}$ is a unit vector.

The angular speed is a constant and is given by

$$\omega=\frac{d\theta}{dt} \tag{4}$$

so the length of the red arc is $\omega\ dt$. But the red arc is also the change in $\boldsymbol{n}$ over the time dt, so its length is the magnitude of $d\boldsymbol{n}/dt$. In the limit as all the differentials go to zero, the direction of the red arc (which is the direction of $d\boldsymbol{n}/dt$) becomes parallel to the radius and pointing inward so the direction of $d\boldsymbol{n}/dt$ points towards the centre of the circle, and its magnitude is ω. Thus we have from 2 and 3

$$\frac{d\boldsymbol{n}}{ds}=\frac{1}{r\omega}\omega\left(-\hat{\boldsymbol{r}}\right)=-\frac{\hat{\boldsymbol{r}}}{r} \tag{5}$$

where $\hat{\boldsymbol{r}}$ is an outward pointing unit vector along the radius.

(b). Now consider the actual velocity vector of the particle. We can use the same figure as above, but now we interpret the blue vectors as the actual velocity vectors, rather than unit vectors. As before, in a time dt the particle moves through an angle $d\theta=\omega\ dt$. The change in the velocity is now given by the red arc on the RHS of the figure, and has magnitude

$$dv=v\ d\theta=v\omega\ dt \tag{6}$$

Thus the magnitude of the acceleration is, using 2

$$a=\frac{dv}{dt}=v\omega=\frac{v^2}{r} \tag{7}$$

Rearranging this, we have

$$r=\frac{v^2}{a} \tag{8}$$

and since the acceleration always points radially inward, we have for the vector pointing from the particle to the centre of the orbit, which is $-\boldsymbol{r}$:

$$-\boldsymbol{r}=\frac{v^{2}}{a}\frac{\boldsymbol{a}}{a}=\frac{v^{2}}{a^{2}}\boldsymbol{a}=\frac{v^{2}}{a}\hat{\boldsymbol{a}} \tag{9}$$

where

$$\hat{\boldsymbol{a}}\equiv\frac{\boldsymbol{a}}{a} \tag{10}$$

is a unit vector in the direction of the acceleration, that is, pointing towards the centre of the circle.