TENSOR SLOT-NAMING NOTATION

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References: Kip S. Thorne & Roger D. Blandford, *Modern Classical Physics*, Princeton University Press (2017). Exercise 1.5.

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T&B refer to tensor components as *slot-naming notation*. In geometric (coordinate-free) form, a tensor of rank n is represented by a letter name with an argument consisting of n slots, into each of which a vector can be inserted. When a vector is inserted into, say, slot 1, then in component notation, the tensor is contracted in slot 1 with the vector that was inserted into that slot. Thus, for example

$$\mathsf{S}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}) = S_{ijk}A_iB_jF_k \tag{1}$$

Ex 1.5(a) Using the tensor product, we have

$$A_iB_{jk} \to \mathsf{A}(\_)\otimes\mathsf{B}(\_,\_) \tag{2}$$

If we contract one of the slots of B with a vector $\boldsymbol{A}$, we have

$$A_iB_{ji} \to \mathsf{B}(\_,\boldsymbol{A}) \tag{3}$$

If a rank-3 tensor possesses the symmetry

$$S_{ijk} = S_{kji} \tag{4}$$

we could write this by labelling the blank slots, as T&B do at the bottom of page 18. It's probably clearer to write

$$\mathsf{S}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) = \mathsf{S}(\boldsymbol{C},\boldsymbol{B},\boldsymbol{A}) \tag{5}$$

Writing this out fully in component form gives

$$S_{ijk}A_iB_jC_k = S_{ijk}C_iB_jA_k \tag{6}$$

Since all indices are summed, we can rename the indices on the RHS by swapping $i \leftrightarrow k$ to give

$$S_{ijk}A_iB_jC_k = S_{kji}C_kB_jA_i \tag{7}$$

If this is to be true for any vectors $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$, then 4 must be satisfied.

Finally, we have the component equation

$$A_i B_i = A_i B_j \mathsf{g}_{ij} \tag{8}$$

where $\mathsf{g}(\_,\_)$ is a rank-2 tensor (the metric tensor). This equation in geometric form is then

$$\boldsymbol{A}(\boldsymbol{B}) = \boldsymbol{B}(\boldsymbol{A}) = \mathsf{g}(\boldsymbol{A},\boldsymbol{B}) \tag{9}$$

Ex 1.5 (b) Given the geometric quantity, we can convert to slot-naming notation.

$$\mathsf{T}(\_,\_,\boldsymbol{A}) \to A_k T_{ijk} \tag{10}$$

For a more complex example, we start on the inside and work towards the outside. We have

$$\mathsf{T}(\_,\mathsf{S}(\boldsymbol{B},\_),\_) \to \mathsf{T}\left(\_,S_{\ell j}B_\ell,\_\right) \tag{11}$$

$$\to T_{ijk}S_{\ell j}B_\ell \tag{12}$$

The quantity $\mathsf{S}(\boldsymbol{B},\_)$ is a rank-2 tensor contracted with a vector, so it leaves one index free, meaning that $\mathsf{S}(\boldsymbol{B},\_)$ behaves like a vector and can be inserted into a slot in another tensor, in this case into T.