KEPLER'S FIRST LAW REVISITED

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.1, problem 2.

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We've shown that the orbit of a planet is elliptical by comparing the solution of Newton's equations with the polar form of an ellipse. In his chapter I.1 on Newton's laws, Zee solves Newton's equations for the central force of gravity in a 'brute force' way and arrives at the equation

$$\frac{1}{2}\dot{r}^2+\frac{\ell^2}{2r^2}-\frac{\kappa}{r}=\epsilon \tag{1}$$

where ℓ and ϵ are constants of integration (which of course turn out to be the angular momentum and total energy per unit mass, respectively) and $\kappa\equiv GM$. This equation can be solved to give r as a function of time, but to show that the orbit is elliptical, it's more useful to have r as a function of θ, the polar angle. To this end, we'd like to convert 1 into a differential equation in r and θ and eliminate the time. To do this, we can use Zee's equation 15:

$$\dot{\theta}=\frac{\ell}{r^2} \tag{2}$$

and change variables to $u\equiv\frac{1}{r}$. Then we get

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \tag{3}$$

$$= \frac{dr}{d\theta}\frac{\ell}{r^2} \tag{4}$$

$$= \frac{d(1/u)}{d\theta}\ell u^2 \tag{5}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\ell u^2 \tag{6}$$

$$= -\ell\frac{du}{d\theta} \tag{7}$$

So 1 becomes

$$\frac{\ell^2}{2}\left(\frac{du}{d\theta}\right)^2+\frac{\ell^2}{2}u^2-\kappa u=\epsilon \tag{8}$$

We can separate the variables to get

$$\frac{\ell du}{\sqrt{2\epsilon+2\kappa u-\ell^2u^2}}=d\theta \tag{9}$$

We can do the integral using Maple, but if you want to do it by hand, complete the square of the quadratic inside the square root:

$$2\epsilon+2\kappa u-\ell^2u^2=C-(A+Bu)^2 \tag{10}$$

$$C-A^2=2\epsilon \tag{11}$$

$$-2AB=2\kappa \tag{12}$$

$$-B^2=-\ell^2 \tag{13}$$

So we have

$$B=\ell \tag{14}$$

$$A=-\frac{\kappa}{\ell} \tag{15}$$

$$C=2\epsilon+\frac{\kappa^2}{\ell^2} \tag{16}$$

$$\frac{\left(\ell/\sqrt{2\epsilon+\frac{\kappa^2}{\ell^2}}\right)du}{\sqrt{1-\left(\ell u-\frac{\kappa}{\ell}\right)^2/\left(2\epsilon+\frac{\kappa^2}{\ell^2}\right)}}=d\theta \tag{17}$$

The integral of the LHS thus gives an arcsine, so

$$\arcsin\left[\frac{\ell\left(u-\frac{\kappa}{\ell^2}\right)}{\sqrt{2\epsilon+\frac{\kappa^2}{\ell^2}}}\right]=\theta+\theta_0 \tag{18}$$

$$u(\theta)=\frac{1}{\ell^2}\left[\sqrt{2\epsilon\ell^2+\kappa^2}\sin(\theta+\theta_0)+\kappa\right] \tag{19}$$

$$r(\theta)=\frac{\ell^2}{\sqrt{2\epsilon\ell^2+\kappa^2}\sin(\theta+\theta_0)+\kappa} \tag{20}$$

$$=\frac{\ell^2}{\sqrt{2\epsilon\ell^2+(GM)^2}\sin(\theta+\theta_0)+GM} \tag{21}$$

If we pick $\theta_0 = \frac{\pi}{2}$, then $\sin(\theta+\theta_0) = \cos\theta$ and we can compare this with the expression we got earlier using the notation of Carroll & Ostlie:

$$r = \frac{L^2/\mu^2}{GM(1+e\cos\theta)} \tag{22}$$

we note that $L/\mu = \ell$ and $\mu = Mm/(M+m)$ is the reduced mass of the planet-sun system. e is the eccentricity of the ellipse. We can shift the origin of the angle θ by $\frac{\pi}{2}$ to convert the sine into the cosine, which leaves us with

$$e = \sqrt{\frac{2\epsilon\ell^2}{G^2M^2}+1} \tag{23}$$

In a bound system, the energy $\epsilon < 0$, so $e < 1$ as required for an ellipse. We get a circular orbit ($e = 0$) when

$$\epsilon_{circ} = -\frac{G^2M^2}{2\ell^2} \tag{24}$$

which is also the minimum of the potential energy part of 1; that is it's the minimum of $\frac{\ell^2}{2r^2} - \frac{\kappa}{r}$.

In his back-of-the-book solution, Zee says that we can recognize 8 as the shifted harmonic oscillator and solve it 'instantly'. I guess I'm not familiar enough with shifted harmonic oscillators to see this instantly, but using completing the square, we can rewrite 8 as ($u' \equiv \frac{du}{d\theta}$):

$$\left(u'\right)^2 + u^2 - \frac{2\kappa}{\ell^2}u = \frac{2\epsilon}{\ell^2} \tag{25}$$

$$\left(u'\right)^2 + \left(u - \frac{\kappa}{\ell^2}\right)^2 = \frac{2\epsilon}{\ell^2} + \frac{\kappa^2}{\ell^4} \tag{26}$$

With $x \equiv u - \frac{\kappa}{\ell^2}$ this equation becomes

$$\frac{1}{2}\left(x'\right)^2 + \frac{1}{2}x^2 = \frac{\epsilon}{\ell^2} + \frac{\kappa^2}{2\ell^4} \tag{27}$$

which is indeed the energy equation for a harmonic oscillator of unit mass and spring constant $k = 1$, with energy $E = \frac{\epsilon}{\ell^2} + \frac{\kappa^2}{2\ell^4}$. Thus the solution is

$$x = A\cos\theta \tag{28}$$

where A is the amplitude of the oscillation, which can be found by setting $\theta = 0$ at which point $x' = 0$ and x has its maximum value of

$$x_{max} = A = \frac{\kappa}{\ell^2}\sqrt{\frac{2\epsilon\ell^2}{\kappa^2}+1} \tag{29}$$

Therefore

$$u = x + \frac{\kappa}{\ell^2} \tag{30}$$

$$= \frac{\kappa}{\ell^2}\left(1 + \sqrt{\frac{2\epsilon\ell^2}{\kappa^2} + 1}\cos\theta\right) \tag{31}$$

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