DEFLECTION OF LIGHT IN NEWTONIAN GRAVITY

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.1, problem 3.

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As we've seen, general relativity using the Schwarzschild metric predicts that light rays are bent as they pass near to a massive body such as the Sun. Newton believed that light was made up of what he called 'corpuscles', or small particles of matter which he presumably believed had mass. Thus in Newton's gravitational theory, light would also be bent as it passed near a massive body.

To get an idea of how much a light corpuscle would be bent, we can assume that it is in a path with an energy per unit mass $\epsilon > 0$, that is, its orbit is unbound. Our previous derivation is valid for $\epsilon > 0$ as well, so we have

$$u(\theta) = \frac{1}{r} = \frac{\kappa}{\ell^2}\left(1+\sqrt{\frac{2\epsilon\ell^2}{\kappa^2}+1}\cos\theta\right) \tag{1}$$

where $\kappa = GM$ and ℓ is the angular momentum per unit mass. This result is a solution of the equations

$$\frac{1}{2}\dot{r}^2 + \frac{\ell^2}{2r^2} - \frac{\kappa}{r} = \epsilon \tag{2}$$

$$\dot{\theta} = \frac{\ell}{r^2} \tag{3}$$

In an unbound orbit, $r \to \infty$ at either end of the orbit meaning $u \to 0$, so the angles $\theta_\pm$ at which this occurs are, from 1:

$$\theta_\pm = \pm\arccos\left[-\left(\frac{2\epsilon\ell^2}{\kappa^2}+1\right)^{-1/2}\right] \tag{4}$$

This problem is essentially one of Rutherford scattering, except the force is due to gravity rather than electrostatics and is attractive rather than repulsive. However, it's usual to express the result in terms of the *impact parameter* b, which is the distance of closest approach to the mass M that the particle would make if there were no force on it.

The angular momentum ℓ is a constant of the motion, so we can work it out when $r \to \infty$. In this limit

$$\ell = pr\sin\theta \tag{5}$$
$$= pb \tag{6}$$

where p is the linear momentum per unit mass. For $r \to \infty$ all the motion is in the r direction (so that $\dot{\theta} \to 0$), so $p \to \dot{r}$ and from 2

$$p = \sqrt{2\epsilon} \tag{7}$$
$$\ell = b\sqrt{2\epsilon} \tag{8}$$

From 4

$$\theta_{\pm} = \pm\arccos\left[-\left(\frac{4\epsilon^2 b^2}{\kappa^2}+1\right)^{-1/2}\right] \tag{9}$$

For weak gravitational fields, we expect κ to be fairly small, so we can expand this as a series around $\kappa = 0$ (I used Maple for this, but you can grind through the derivatives if you like) and get

$$\theta_{\pm} = \pm\left[\frac{\pi}{2}+\frac{\kappa}{2b\epsilon}-\frac{\kappa^3}{24b^3\epsilon^3}+\mathcal{O}\left(\kappa^5\right)\right] \tag{10}$$

First, we note that if there is no mass, $\kappa = GM = 0$ and $\theta_{\pm} = \pm\frac{\pi}{2}$. That is, the angular difference between the incoming and outgoing photon is π which is a straight line with no deflection as we'd expect. As we increase the mass, the deflection from a straight line is, to first order in κ:

$$\Delta\theta = \theta_{+} - \theta_{-} - \pi \tag{11}$$
$$= \frac{\kappa}{b\epsilon} \tag{12}$$

The deflection increases with mass, but decreases with increasing photon energy (in Newtonian physics, 'increasing photon energy' means the photon is going faster; relativity was unknown at the time) and with increasing impact parameter (the larger the closest approach to the mass, the smaller the deflection). This all seems to make sense.

From 2 at $r \to \infty$ and taking $\dot{r} = c$, the speed of light, we get $\epsilon = \frac{c^2}{2}$ so the deflection becomes

$$\Delta\theta = \frac{2GM}{bc^2} \tag{13}$$