GRAVITATIONAL FORCE DUE TO A SPHERE AND THE GRAVITY EXPRESS TRAIN

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.1, problem 4 - 6.

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Zee refers to a couple of Newton's results on gravity as his two 'superb theorems'. The first theorem states that the gravitational force due to a spherically symmetric mass distribution is equal to the force due to a point mass at the centre of the sphere. The second theorem states that the gravitational force inside a spherical shell of mass (of uniform density) is zero.

Although we *could* prove these theorems by doing elaborate integrals over spherical shells of mass, it is easiest to note that because Newton's gravitational force obeys an inverse square law, it is mathematically equivalent to the electric force due to a static charge distribution, so we can apply Gauss's law to the gravitational case. To do this, we make the substitution

$$\mathbf{E} \to \mathbf{g} \tag{1}$$

$$\frac{1}{4\pi\epsilon_0} \to G \tag{2}$$

where **g** is the gravitational field. Gauss's law for Newtonian gravity is therefore (in integral and differential forms):

$$\int \mathbf{g}\cdot d\mathbf{a} = 4\pi G\int \rho(\mathbf{r})\, d^3\mathbf{r} \tag{3}$$

$$\nabla\cdot\mathbf{g} = 4\pi G\rho(\mathbf{r}) \tag{4}$$

The integral extends over a Gaussian volume as usual.

For a spherically symmetric mass distribution, **g** is radial (by symmetry), so at a distance r from the centre of the sphere

$$\int \mathbf{g}\cdot d\mathbf{a} = 4\pi r^2 g \tag{5}$$

$$= 4\pi GM \tag{6}$$

$$g = \frac{GM}{r^2} \tag{7}$$

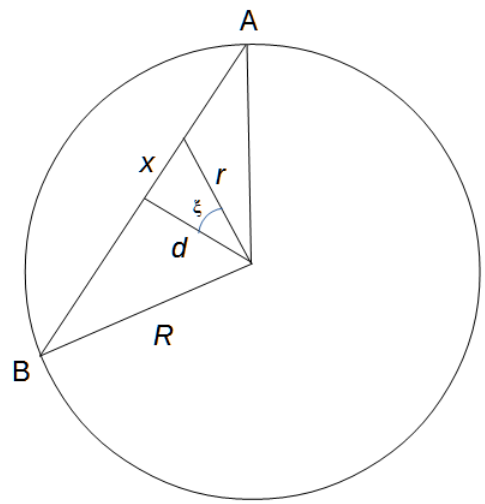

FIGURE 1. The gravity express train.

That is, the field is equivalent to a point mass M at a distance r.

For a point inside a spherical shell, again **g** is radial (by symmetry) and there is no enclosed charge so $\rho = 0$. Therefore $\mathbf{g} = 0$ and there is no field.

An example of these two theorems is the 'gravity express train' that is a standard of pretty well every first year physics course. Suppose we can build a tunnel connecting two cities A and B as shown in Fig. 1.

If we can run a frictionless train through this tunnel, the force of gravity gives us a free ride between the two cities. Suppose the train is in the tunnel at a distance r from the Earth's centre, as shown. At this point, only the mass of the sphere of radius r exerts any force on the train, and it is a radial force directed towards the centre of the Earth (with radius R and mass M). The component of this force parallel to the tunnel will pull on the train, so the force on the train (with mass m) is

$$F = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right)\frac{GMm}{r^2}\sin\xi \tag{8}$$

$$= \frac{GMm}{R^3}(r\sin\xi) \tag{9}$$

$$= \frac{GMm}{R^3}x \tag{10}$$

where $x = r\sin\xi$ as in the figure. That is, the force is of the form $F = kx$ which results in simple harmonic motion, with angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM}{R^3}} \tag{11}$$

The period of the oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{12}$$

Plugging in the numbers gives

$$T = 2\pi\sqrt{\frac{(6.371\times 10^6)^3}{(6.67\times 10^{-11})(5.972\times 10^{24})}} \tag{13}$$

$$= 5062 \text{ sec} \tag{14}$$

$$= 84.4 \text{ min} \tag{15}$$

So we'd get from A to B in around 42 minutes. Notice that this result is independent of d, so that it doesn't matter where the two cities are located. They could be on opposite sides of the Earth, or separated by only a few kilometres; the travel time is the same. Of course the engineering required to build such a tunnel is beyond our current technology (remember how long it took just to build the Channel Tunnel between England and France), but it's a nice idea.