CONSERVATION OF LINEAR AND ANGULAR MOMENTUM IN NEWTONIAN MECHANICS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.2, problem 1.

Post date: 6 Apr 2020.

The nature of the dependence of a force or potential on the underlying position coordinates can determine certain conservation laws. In his chapter I.2, Zee shows that a central force (a force that is always directed towards its source, such as the Earth's gravity or a point charge's electrostatic field) conserves angular momentum. Actually his derivation is a generalization to any number $D \geq 2$ dimensions of the more familiar proof in 3-d, which goes like this:

The angular momentum of a mass m is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1}$$

where $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$ is the linear momentum. Taking the time derivative we get

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \tag{2}$$

$$= m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \mathbf{F} \tag{3}$$

$$= 0 + 0 \tag{4}$$

$$= 0 \tag{5}$$

where the third line uses the fact that $\mathbf{r} \parallel \mathbf{F}$ for a central force, so their cross product is zero. Thus $\mathbf{L}$ doesn't change with time.

Now suppose that a force F is the negative gradient of a potential function V so that by Newton's law:

$$m_a \frac{d^2 x_a^i}{dt^2} = -\frac{\partial V(x)}{\partial x_a^i} \tag{6}$$

where the index a refers to particle a in a collection of N interacting particles, and i is the component of the coordinate x. Note that the x in $V(x)$ represents all D components of x (if we're doing the calculation in D-dimensional space) and not just the magnitude of the distance.

Now suppose that V is a function only of the coordinate *differences* $x_a^i - x_b^i$ between particles a and b, where $a, b = 1, \ldots, N$ with $a \neq b$. In this case, the total linear momentum is given by

$$p^i = \sum_a m_a \frac{dx_a^i}{dt} \tag{7}$$

The time derivative is

$$\dot{p}^i = \sum_a m_a \frac{d^2 x_a^i}{dt^2} \tag{8}$$

$$= -\sum_a \frac{\partial V(x)}{\partial x_a^i} \tag{9}$$

Since V is a function of the set of all possible differences $x_a^i - x_b^i$, the terms in the sum 9 cancel in pairs. For example, for 3 particles if $V = f\left(\left(x_a^i - x_b^i\right), \left(x_a^i - x_c^i\right), \left(x_b^i - x_c^i\right)\right)$, then $\frac{\partial V}{\partial x_a^i}$ will contain a term equivalent to $\frac{\partial V}{\partial\left(x_a^i - x_b^i\right)}$ (plus another term resulting from $\frac{\partial V}{\partial(x_a^i - x_c^i)}$). However, $\frac{\partial V}{\partial x_b^i}$ will contain a term equivalent to $-\frac{\partial V}{\partial\left(x_a^i - x_b^i\right)}$ which cancels the first term. That is

$$\frac{\partial V}{\partial x_a^i} = \frac{\partial f}{\partial\left(x_a^i - x_b^i\right)} + \frac{\partial f}{\partial\left(x_a^i - x_c^i\right)} \tag{10}$$

$$\frac{\partial V}{\partial x_b^i} = -\frac{\partial f}{\partial\left(x_a^i - x_b^i\right)} + \frac{\partial f}{\partial\left(x_b^i - x_c^i\right)} \tag{11}$$

$$\frac{\partial V}{\partial x_c^i} = -\frac{\partial f}{\partial\left(x_a^i - x_c^i\right)} - \frac{\partial f}{\partial\left(x_b^i - x_c^i\right)} \tag{12}$$

so adding up all the derivatives causes the terms to cancel in pairs. The argument is fairly easily extended to N particles.

Thus $\dot{p}^i = 0$ and linear momentum is conserved. [I realize this isn't a very elegant way of proving it; there is probably a better way of writing it down, but hopefully you get the idea.]