VECTORS AND NON-VECTORS IN 3-D ROTATIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.3, Problem 1.

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When you first meet vectors in a linear algebra course, it's easy to get the impression that a vector is (in 3-d) just 3 numbers placed in a column, like

$$\vec{a}=\left[\begin{array}{c}1\\2\\3\end{array}\right] \tag{1}$$

However, as used by Zee, a vector is a specific type of object that transforms under a rotation according to

$$\vec{a}^{\prime}=R\vec{a} \tag{2}$$

where R is a 3-d rotation matrix. One consequence of this is that if we start with a vector $\vec{p}$ (that is, an object that *does* transform properly under a rotation), then we can't just take the components of $\vec{p}$ and combine them in an arbitrary way to create another object that is a column of 3 numbers, expecting that new object to be a vector in the sense just defined. That is, the new object probably won't transform properly under a rotation.

For example, suppose we have 2 vectors $\vec{p}$ and $\vec{q}$ that do transform properly under a rotation. This means that (using the summation convention)

$$p^{\prime i}=R^{ij}p^{j} \tag{3}$$

$$q^{\prime i}=R^{ij}q^{j} \tag{4}$$

Now consider the array

$$A=\left[\begin{array}{c}p^{2}q^{3}\\p^{3}q^{1}\\p^{1}q^{2}\end{array}\right] \tag{5}$$

Is this a vector? To check, we must use the transformation equations above for the components of $\vec{p}$ and $\vec{q}$, since we know how these transform. We get, for the first component:

$$A'^{1} = p'^{2} q'^{3} = R^{2j} p^{j} R^{3k} q^{k} \tag{6}$$

In order for this to transform properly, we would need this to be equal to $R^{1m} A^{m}$, that is

$$R^{1m} A^{m} = R^{11} p^{2} q^{3} + R^{12} p^{3} q^{1} + R^{13} p^{1} q^{2} \tag{7}$$

In order for that to be true, we'd need

$$R^{22} R^{33} = R^{11} \tag{8}$$
$$R^{23} R^{31} = R^{12} \tag{9}$$
$$R^{21} R^{33} = R^{13} \tag{10}$$

This isn't true as can be seen by looking at a rotation about the x axis:

$$R_{x}(\theta_{x}) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\theta_{x} & \sin\theta_{x} \\ 0 & -\sin\theta_{x} & \cos\theta_{x} \end{array}\right) \tag{11}$$

In this case, for example, $R^{22} R^{33} = \cos^{2}\theta_{x} \neq R^{11} = 1$.

We can show that the vector cross product $\vec{p} \times \vec{q}$ *does* transform properly under a general 3-d rotation. In this case

$$\vec{p} \times \vec{q} = \left[\begin{array}{c} p^{2} q^{3} - p^{3} q^{2} \\ p^{3} q^{1} - p^{1} q^{3} \\ p^{1} q^{2} - p^{2} q^{1} \end{array}\right] \tag{12}$$

Transforming the first component by transforming $\vec{p}$ and $\vec{q}$ separately first gives

$$\left(p^{2} q^{3} - p^{3} q^{2}\right)' = R^{2j} R^{3k} p^{j} p^{k} - R^{2k} R^{3j} p^{j} q^{k} \tag{13}$$
$$= p^{j} q^{k} \left(R^{2j} R^{3k} - R^{2k} R^{3j}\right) \tag{14}$$

Treating $\vec{p} \times \vec{q}$ as a vector and transforming it directly gives

$$(\vec{p} \times \vec{q})'^{1} = R^{11}\left(p^{2} q^{3} - p^{3} q^{2}\right) + R^{12}\left(p^{3} q^{1} - p^{1} q^{3}\right) + R^{13}\left(p^{1} q^{2} - p^{2} q^{1}\right) \tag{15}$$

Comparing this with 14 shows that the two expressions are equal if

$$\begin{aligned} R^{11} &= R^{22}R^{33} - R^{23}R^{32} \\ R^{12} &= R^{31}R^{23} - R^{21}R^{33} \\ R^{13} &= R^{21}R^{32} - R^{31}R^{22} \end{aligned} \tag{16}$$

The terms in 14 with $j = k$ are all zero.

To verify these equations, we need to recall the two properties that R must satisfy: $\det R = 1$ and $R^T = R^{-1}$. The determinant condition gives us (expanding about the first row):

$$R^{11}\left(R^{22}R^{33} - R^{23}R^{32}\right) - R^{12}\left(R^{21}R^{33} - R^{31}R^{23}\right) + R^{13}\left(R^{21}R^{32} - R^{31}R^{22}\right) = 1 \tag{17}$$

The condition $R^T = R^{-1}$ gives

$$R^{ij}R^{kj} = \delta^{ik} \tag{18}$$

so for $i = k = 1$ we have

$$\left(R^{11}\right)^2 + \left(R^{12}\right)^2 + \left(R^{13}\right)^2 = 1 \tag{19}$$

This is the equation of a sphere of radius 1 in a rectangular coordinate system with coordinates R^{11}, R^{12} and R^{13}. Equation 17 is the equation of a plane in the same coordinate system, so to satisfy both equations means taking the intersection of the plane and the sphere. This requires equations 16 to be true, so the cross product is indeed a proper vector that transforms correctly under rotation.

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