LIE'S METHOD OF GENERATING ROTATIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.3, Problem 3.

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As an introduction to the idea of invariance under a coordinate transformation, Zee looks at rotations in detail in his chapter I.3. He does most of the derivation in 2 dimensions, so we'll summarize his results, but use 3 dimensions to illustrate.

A vector is defined as an object that transforms like the rectangular coordinates under rotation, that is for a vector $\vec{p}$, its form $\vec{p}'$ in a rotated system is given by the matrix equation

$$\vec{p}' = R(\theta)\,\vec{p} \tag{1}$$

where R is the rotation matrix and θ is the angle of rotation. In 2 dimensions, we're assuming that θ is a counterclockwise rotation about the origin; in higher dimensions, we need to specify the axis of rotation, so there will be different R matrices for different axes.

The square of a vector (equivalent to the square of its length) must remain unchanged by a rotation, so that

$$\vec{p}'^{T} \cdot \vec{p}' = \vec{p}^{T} \cdot \vec{p} \tag{2}$$

where T denotes the transpose of the vector (converting a column vector into a row vector).

Since this holds for all vectors, it also holds for a sum of two vectors (which is also a vector), so:

$$\left(\vec{u}'^{T} + \vec{v}'^{T}\right) \cdot \left(\vec{u}' + \vec{v}'\right) = \left(\vec{u}^{T} + \vec{v}^{T}\right) \cdot \left(\vec{u} + \vec{v}\right) \tag{3}$$

$$\vec{u}'^{T} \cdot \vec{v}' = \vec{u}^{T} \cdot \vec{v} \tag{4}$$

where the last line comes from multiplying out the first line and applying 2, and using the fact that $\vec{u}^{T} \cdot \vec{v} = \vec{v}^{T} \cdot \vec{u}$. Thus the scalar or dot product is also an invariant under rotation. This gives us a condition on R, since

$$\vec{u}'^T \cdot \vec{v}' = (R\vec{u})^T (R\vec{v}) \tag{5}$$

$$= \vec{u}^T \left(R^T R\right) \vec{v} \tag{6}$$

$$R^T R = I \tag{7}$$

where I is the identity matrix.

At this point, most introductory courses use a bit of geometry and trigonometry to arrive at the rotation matrix R in terms of sines and cosines. Zee takes an alternative approach by introducing Lie's (pronounced "Lee") method of deriving the rotation matrix. Lie starts by considering an infinitesimal rotation. Since no rotation at all amounts to $R = I$ (no change in any vector), we can write

$$R = I + A \tag{8}$$

for an infinitesimal rotation, where A is a matrix that contains only infinitesimal elements (so that A^n, where $n \geq 2$, terms can be neglected). From 7 we get

$$R^T R = (I + A)^T (I + A) \tag{9}$$

$$= I + A^T + A + \mathcal{O}\left(A^2\right) \tag{10}$$

$$= I \tag{11}$$

Therefore, A must be antisymmetric:

$$A^T = -A \tag{12}$$

An antisymmetric matrix always has only zeroes on the diagonal, so the only elements that can be specified independently are those in the upper (or lower) triangle. In D dimensions, there are $\sum_{n=1}^{D-1} n = \frac{1}{2} D (D-1)$ such elements. Thus in 2 dimensions, there is only 1 element, in 3 dimensions there are 3, in 4 dimensions there are 6 and so on.

Looking at 3 dimensions, the 3 independent antisymmetric matrices are

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{13}$$

$$\mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{14}$$

$$\mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

These 3 matrices are known as the generators of the 3-d rotation group $SO(3)$.

Any 3-d antisymmetric matrix A can be written as a linear combination of these 3 matrices:

$$A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z \tag{16}$$

Let's consider a rotation about the x axis, so that $\theta_y = \theta_z = 0$. If we want to rotate through some *finite* (not infinitesimal) angle θ_x we can do it by applying lots of infinitesimal rotations. For some large number N, we can rotate through θ_x by applying N rotations of size θ_x/N. That is, the rotation matrix R for a finite rotation is, in the limit,

$$R(\theta_x) = \lim_{N\to\infty} \left[R\left(\frac{\theta_x}{N}\right) \right]^N \tag{17}$$

$$= \lim_{N\to\infty} \left[I + \frac{\theta_x \mathcal{J}_x}{N} \right]^N \tag{18}$$

This limit turns out to be one of the definitions of the exponential function, that is

$$e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N \tag{19}$$

This can be proved by taking the derivative:

$$\frac{d}{dx}e^x = \lim_{N\to\infty}\left[N\frac{1}{N}\left(1+\frac{x}{N}\right)^{N-1}\right] \tag{20}$$

$$= \lim_{N\to\infty}\left[\left(1+\frac{x}{N}\right)^{-1}\left(1+\frac{x}{N}\right)^{N}\right] \tag{21}$$

$$= \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^{N} \tag{22}$$

$$= e^x \tag{23}$$

Thus we can write 18 as

$$R(\theta_x) = e^{\theta_x \mathcal{J}_x} \tag{24}$$

The exponential of a matrix can be evaluated by writing it as a Taylor series for the exponential:

$$e^x = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} \tag{25}$$

[I've separated out the 1 to avoid problems when $x = 0$, for which we'd get 0^0 in the sum if we started at $n = 0$.]

This might not seem to get us anywhere, since we're faced with an infinite series of powers of the matrix $\mathcal{J}_x$. However, there are only 4 distinct values of these powers:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{26}$$

$$\mathcal{J}_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{27}$$

$$\mathcal{J}_x^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -\mathcal{J}_x \tag{28}$$

$$\mathcal{J}_x^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\mathcal{J}_x^2 \tag{29}$$

After this, the powers repeat themselves in cycles of 4. Separating the odd an even powers, we get

$$e^{\theta_x \mathcal{J}_x} = I + \sum_{k=1}^{\infty} (-1)^k \frac{\theta_x^{2k}}{(2k)!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta_x^{2k-1}}{(2k-1)!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{30}$$

The sums apply only to the lower right 2×2 submatrix, for which we can use the series expansions of sine and cosine:

$$\sin x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \tag{31}$$

$$\cos x = 1 + \sum_{k=1}^{\infty} (-1)^{k} \frac{x^{2k}}{(2k)!} \tag{32}$$

We therefore get

$$R_x(\theta_x) = e^{\theta_x \mathcal{J}_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \tag{33}$$

By the same sort of calculation, we can get the rotation matrix about the y axis:

$$R_y(\theta_y) = e^{\theta_y \mathcal{J}_y} \tag{34}$$

$$= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \tag{35}$$

[Note the sign convention: the negative term $-\sin\theta_y$ is in the upper right rather than the lower left. This is true for a right-handed xyz system.]

Note that rotations about the x and y axes don't commute:

$$R_x(\theta_x) R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ \sin\theta_x \sin\theta_y & \cos\theta_x & \sin\theta_x \cos\theta_y \\ \cos\theta_x \sin\theta_y & -\sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix} \tag{36}$$

$$R_y(\theta_y) R_x(\theta_x) = \begin{pmatrix} \cos\theta_y & \sin\theta_x \sin\theta_y & -\sin\theta_y \cos\theta_x \\ 0 & \cos\theta_x & \sin\theta_x \\ \sin\theta_y & -\sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y \end{pmatrix} \tag{37}$$

$$\neq R_x(\theta_x) R_y(\theta_y) \tag{38}$$

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