LIE ROTATIONS IN HIGHER DIMENSIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.3, Problem 4.

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We can generalize Lie's method of generating rotations to any number D of dimensions. [This post follows Appendix 2 in Zee's chapter I.3, which I believe contains a few typos. I'll try to correct them here.]

A rotation should leave the dot product of two vectors unchanged. If we consider an infinitesimal rotation given by the matrix $R = I + A$, where I is the identity matrix and A is a matrix containing infinitesimal quantities, then the invariance of the dot product leads to the conclusion that $R^T R = I$ which requires (to first order in A) that $A^T = -A$, that is, that A is antisymmetric. In D dimensions, there are $\frac{1}{2}D(D-1)$ independent antisymmetric matrices, which can be written down by choosing a row m and a column $n < m$ (since diagonal elements are all zero in an antisymmetric matrix), setting the element $\mathcal{J}^{mn} = 1$ and the element $\mathcal{J}^{nm} = -1$. Then any antisymmetric matrix A can be decomposed into a linear combination of the $\mathcal{J}$ matrices. To distinguish the $\mathcal{J}$s, we'll given them a subscript (mn) to label which row and column are non-zero. For example in 3-d, we have

$$\mathcal{J}_{(32)} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right) \tag{1}$$

$$\mathcal{J}_{(31)} = \left(\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \tag{2}$$

$$\mathcal{J}_{(21)} = \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{3}$$

[Note that these matrices aren't quite the same as those we used in the last post, since $\mathcal{J}_{(32)} = -\mathcal{J}_x$ and $\mathcal{J}_{(21)} = -\mathcal{J}_z$.] The subscript labels the entire *matrix* and not just a single element within a matrix. To pick out an individual element, we'll use superscript indices, so that, for example, $\mathcal{J}_{(32)}^{23} = -1$ is the element of $\mathcal{J}_{(32)}$ in row 2 and column 3.

Because these matrices are related to the angular momentum operators in quantum mechanics, it's customary to make them into hermitian operators, which, for a matrix, means that the complex conjugate of the transpose (the hermitian conjuage) is the same as the original matrix. That is, for a matrix $J_{(mn)}$ we have $\left(J_{(mn)}^{T}\right)^{*}=J_{(mn)}$. Since the $\mathcal{J}_{(mn)}$ are antisymmetric and real, their hermitian conjugates are antisymmetric, so they aren't hermitian matrices. We can convert them into hermitian matrices by multiplying them by a multiple of $i=\sqrt{-1}$; by convention this multiple is taken to be $-i$. Thus we have the hermitian matrices

$$J_{(mn)}\equiv-i\mathcal{J}_{(mn)} \tag{4}$$

We can write out the $J_{(mn)}$ in a general formula using the Kronecker delta:

$$J_{(mn)}^{ij}=-i\left(\delta^{mi}\delta^{nj}-\delta^{mj}\delta^{ni}\right) \tag{5}$$

To verify this formula, remember that $J_{(mn)}^{mn}=-J_{(mn)}^{nm}=-i$ with all other elements being zero. The first term in the RHS of 5 is non-zero only if $m=i$ and $n=j$, while the second term in the RHS is non-zero only if $m=j$ and $n=i$, so the formula is correct.

[In the paragraph following Zee's equation 19, he says "there are only $\frac{1}{2}D(D-1)$ real antisymmetric D-by-D matrices $J_{(mn)}$". The last $J_{(mn)}$ should be $\mathcal{J}_{(mn)}$ since the $J_{(mn)}$ contain purely imaginary elements, not real ones.]

To generate a rotation in D dimensions, we can use Lie's method of considering infinitesimal rotations $R=I+A$, where A is an infinitesimal linear combination of the $J_{(mn)}$. Since A is real, we have

$$A=i\sum_{i}\theta_{i}J_{i} \tag{6}$$

for some real values θ_i.

The next stage in the argument isn't entirely clear to me, probably because I haven't seen the use to which rotations are put in the rest of Zee's book. However, let's plow on for the moment.

We saw in the previous post that in 3-d, rotations about different axes do not commute; a rotation about x and then y will leave you in a different orientation than a rotation about y and then x. Generalizing to D dimensions, suppose we have two infinitesimal rotations $R_1=I+A+\mathcal{O}\left(A^2\right)$ and $R_2=I+B+\mathcal{O}\left(B^2\right)$, where A and B are infinitesimal, antisymmetric matrices as before. Then if we apply R_2 first, then R_1, the overall rotation is given by

$$R_1R_2 = \left(I + A + \mathcal{O}\left(A^2\right)\right)\left(I + B + \mathcal{O}\left(B^2\right)\right) \tag{7}$$
$$= I + A + B + AB + \mathcal{O}\left(A^2, B^2\right) \tag{8}$$

Switching the order [note that Zee has a typo here: he says $R_2R_1 \simeq (I + A)(I + B)$; it should be $R_2R_1 \simeq (I + B)(I + A)$] we get

$$R_2R_1 = \left(I + B + \mathcal{O}\left(B^2\right)\right)\left(I + A + \mathcal{O}\left(A^2\right)\right) \tag{9}$$
$$= I + B + A + BA + \mathcal{O}\left(A^2, B^2\right) \tag{10}$$

Taking the difference, we get

$$R_1R_2 - R_2R_1 = AB - BA \tag{11}$$
$$= [A, B] \tag{12}$$

where $[A, B] \equiv AB - BA$ is the commutator of A and B (the same commutators that show up in quantum mechanics). This derivation seems to be a bit of a fudge, since the commutator is, by definition, of second order in A and B, so by separating it out from the general $\mathcal{O}\left(A^2, B^2\right)$ term, it seems we're implicitly assuming that the $\mathcal{O}\left(A^2, B^2\right)$ is the same in both R_1R_2 and R_2R_1, so it cancels out when taking the difference.

Zee gives a second argument for measuring the difference between the two compound rotations R_1R_2 and R_2R_1. If the two orders of rotation commuted, then the inverse of one rotation should also be the inverse of the other. That is, we should have $(R_2R_1)^{-1}(R_1R_2) = I$. A measure of how different the two orders of rotation are from each other can then be found by seeing how much $(R_2R_1)^{-1}(R_1R_2)$ differs from I.

For an infinitesimal rotation $R = I + A$, then to first order $R^{-1} = I - A$ since $R^{-1}R = I + A - A - A^2 = I + \mathcal{O}\left(A^2\right)$. Therefore

$$(R_2R_1)^{-1}(R_1R_2) = \left[I - \left(B + A + BA + \mathcal{O}\left(A^2, B^2\right)\right)\right]\left[I + A + B + AB + \mathcal{O}\left(A^2, B^2\right)\right] \tag{13}$$
$$= I + [A, B] - (A + B)^2 + \mathcal{O}\left(A^2, B^2\right) \tag{14}$$

Again, the $(A + B)^2$ term is neglected along with the $\mathcal{O}\left(A^2, B^2\right)$ terms, even though it contains the terms $AB + BA$ which are the same terms that appear in $[A, B]$. It's not clear to me how we can justify ignoring $AB + BA$ but not $[A, B]$.

In any case, we can observe that the transpose of a commutator gives

$$\begin{aligned}[][A,B]^T &= (AB - BA)^T && (15)\\ &= (BA - AB) && (16)\\ &= -[A,B] && (17)\end{aligned}$$

so a commutator is always an antisymmetric matrix. In particular, for the J_i matrices in 5, we have

$$[J_i, J_j] = i c_{ijk} J_k \tag{18}$$

with an implied sum over k on the RHS. This follows because any antisymmetric matrix can be written as a linear combination of the J_ks.

Since the J_is are purely imaginary, a product of two of them is always real. Thus the commutator $[J_i, J_j]$ must be a real matrix. Therefore the c_{ijk} coefficients must be real, since iJ_k is a real matrix.

We can find the c_{ijk} coefficients by a brute force calculation starting with 5. We get (with an implied sum over the index i):

$$\begin{aligned}\left[J_{(mn)}, J_{(pq)}\right]^{k\ell} &= J_{(mn)}^{ki} J_{(pq)}^{i\ell} - J_{(pq)}^{ki} J_{(mn)}^{i\ell} && (19)\\ &= -\left[\delta^{mk}\delta^{ni} - \delta^{mi}\delta^{nk}\right]\left[\delta^{pi}\delta^{q\ell} - \delta^{qi}\delta^{p\ell}\right] && (20)\\ &\quad + \left[\delta^{pk}\delta^{qi} - \delta^{pi}\delta^{qk}\right]\left[\delta^{mi}\delta^{n\ell} - \delta^{ni}\delta^{m\ell}\right]\\ &= -\delta^{mk}\delta^{np}\delta^{q\ell} + \delta^{mk}\delta^{nq}\delta^{p\ell} + \delta^{mp}\delta^{nk}\delta^{q\ell} - \delta^{mq}\delta^{nk}\delta^{p\ell} && (21)\\ &\quad + \delta^{pk}\delta^{mq}\delta^{n\ell} - \delta^{mp}\delta^{qk}\delta^{n\ell} - \delta^{pk}\delta^{qn}\delta^{m\ell} + \delta^{np}\delta^{qk}\delta^{m\ell}\\ &= \delta^{np}\left(\delta^{qk}\delta^{m\ell} - \delta^{mk}\delta^{q\ell}\right) + \delta^{nq}\left(\delta^{mk}\delta^{p\ell} - \delta^{m\ell}\delta^{pk}\right) && (22)\\ &\quad + \delta^{mp}\left(\delta^{nk}\delta^{q\ell} - \delta^{qk}\delta^{n\ell}\right) + \delta^{mq}\left(\delta^{pk}\delta^{n\ell} - \delta^{nk}\delta^{p\ell}\right)\\ &= i\left[\delta^{mp}J_{(nq)} + \delta^{nq}J_{(mp)} - \delta^{np}J_{(mq)} - \delta^{mq}J_{(np)}\right]^{k\ell} && (23)\end{aligned}$$

So the commutator is

$$\left[J_{(mn)}, J_{(pq)}\right] = i\left[\delta^{mp}J_{(nq)} + \delta^{nq}J_{(mp)} - \delta^{np}J_{(mq)} - \delta^{mq}J_{(np)}\right] \tag{24}$$

which agrees with Zee's eqn 24.

For general infinitesimal rotations from 6

$$A = i\sum_i \theta_i J_i \tag{25}$$

$$B = i\sum_j \theta'_j J_j \tag{26}$$

The commutator is therefore

$$[A,B] = i^2\left[\sum_{i,j}\theta_i\theta'_j J_i J_j - \sum_{i,j}\theta_i\theta'_j J_j J_i\right] \tag{27}$$

$$= -\sum_{i,j}\theta_i\theta'_j\,[J_i, J_j] \tag{28}$$

Thus if we know the commutators of the generator matrices J_i we can work out the commutators of any antisymmetric matrix pair.