# AVERAGE OF A VECTOR OVER ALL DIRECTIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.3, Problem 5.

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In this problem, Zee gives us a 3-d vector $\vec{p}$ and asks us to find the quantity $p^i p^j$ averaged over the direction of $\vec{p}$. I wasn't entirely clear what this question was asking, since once you've defined $\vec{p}$, surely its direction is fixed so how can you average over a fixed direction? I think what he means is: suppose you rotate $\vec{p}$ through all possible angles in 3-d while keeping its magnitude $|\vec{p}|$ fixed. What then is the average of $p^i p^j$ over all these rotations?

There is also a mistake in the formula he gives for the average. If $\theta$ and $\phi$ are the usual spherical angles, then the average of $p^i p^j$ is given by

$$\left\langle p^i p^j \right\rangle = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} d\theta d\phi \sin\theta p^i p^j \tag{1}$$

(not $\cos\theta$ in the integral).

We can work out the integral by writing $p^i$ in rectangular coordinates in the usual way:

$$p^x = p\sin\theta\cos\phi \tag{2}$$
$$p^y = p\sin\theta\sin\phi \tag{3}$$
$$p^z = p\cos\theta \tag{4}$$

From symmetry, $\left\langle p^i p^j \right\rangle = 0$ if $i \neq j$. If this isn't obvious, suppose we're looking at $\left\langle p^x p^y \right\rangle$, and we pick one direction specified by angles $\theta_1$ and $\phi_1$. Then the initial coordinates are

$$p_1^x = p\sin\theta_1\cos\phi_1 \tag{5}$$
$$p_1^y = p\sin\theta_1\sin\phi_1 \tag{6}$$

If we then rotate $\vec{p}$ by keeping $\theta_1$ constant and rotating $\phi$ by $2\left(\frac{\pi}{2}-\phi_1\right)$ so that $\phi_2 = \phi_1 + 2\left(\frac{\pi}{2}-\phi_1\right) = \pi - \phi_1$ then

$$p_2^x = p\sin\theta_1\cos(\pi-\phi_1) = -p\sin\theta_1\cos\phi_1 = -p_1^x \quad (7)$$
$$p_2^y = p\sin\theta_1\sin(\pi-\phi_1) = p\sin\theta_1\sin\phi_1 = p_1^y \quad (8)$$

Thus this position for $\vec{p}$ cancels the original position in the average, so the overall average $\langle p^x p^y\rangle = 0$. A similar argument applies to $\langle p^x p^z\rangle$ and $\langle p^y p^z\rangle$.

We could also work out these averages by direct integration. For example

$$\langle p^x p^y\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta\,(p\sin\theta\cos\phi)(p\sin\theta\sin\phi) \quad (9)$$
$$= \frac{p^2}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin^3\theta\sin\phi\cos\phi \quad (10)$$
$$= 0 \quad (11)$$

For the other terms, we have

$$\langle p^x p^x\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta\,(p\sin\theta\cos\phi)^2 \quad (12)$$
$$= \frac{p^2}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin^3\theta\cos^2\phi \quad (13)$$
$$= \frac{p^2}{3} \quad (14)$$
$$\langle p^y p^y\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta\,(p\sin\theta\sin\phi)^2 \quad (15)$$
$$= \frac{p^2}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin^3\theta\sin^2\phi \quad (16)$$
$$= \frac{p^2}{3} \quad (17)$$
$$\langle p^z p^z\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta\,(p\cos\theta)^2 \quad (18)$$
$$= \frac{p^2}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta\cos^2\theta \quad (19)$$
$$= \frac{p^2}{3} \quad (20)$$

In summary

$$\langle p^i p^j\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} d\theta d\phi \sin\theta p^i p^j = \frac{p^2}{3}\delta^{ij} \quad (21)$$