TRANSFORMATIONS OF GRADIENT AND LAPLACIAN

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problem 1.

Post date: 8 Apr 2020.

I've done a lot of posts involving tensors, but Zee's approach is a simpler way of looking at them, so it's worth revisiting the definition of a tensor. Zee's definition of a tensor is that it is "something that transforms like a tensor". Of course, in order for this definition to make sense, we need to know how a tensor transforms. If we have a tensor with 2 indices such as T^{ij}, then it must transform under a rotation according to (using the summation convention):

$$T'^{ij}=R^{ik}R^{j\ell}T^{k\ell} \tag{1}$$

[At this stage, we're not distinguishing between upper and lower indices; that will come later.] Thus a vector is a tensor with a single index.

Example 1. The gradient of a scalar is a tensor (actually a vector), as we can see using the following.

$$(\nabla\phi)^i\equiv\frac{\partial\phi}{\partial x^i} \tag{2}$$

If we rotate the coordinate system, then the new coordinates are related to the old ones by

$$x'^i=R^{ij}x^j \tag{3}$$

Thus we have

$$\frac{\partial x'^i}{\partial x^j}=R^{ij} \tag{4}$$

Because the inverse of a rotation matrix is also its transpose, we have

$$x^i=\left(R^{-1}\right)^{ij}x'^j \tag{5}$$

$$=\left(R^T\right)^{ij}x'^j \tag{6}$$

$$=R^{ji}x'^j \tag{7}$$

Therefore

$$\frac{\partial x^i}{\partial x'^j} = R^{ji} \tag{8}$$

Using the chain rule

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} \tag{9}$$

$$= R^{ij}\frac{\partial}{\partial x^j} \tag{10}$$

Therefore the rotation matrix is given by

$$R^{ij} = \frac{\partial x^j}{\partial x'^i} \tag{11}$$

The rotated gradient is thus

$$\left(\nabla'\phi\right)^i = \frac{\partial \phi}{\partial x'^i} \tag{12}$$

$$= \frac{\partial x^j}{\partial x'^i}\frac{\partial \phi}{\partial x^j} \tag{13}$$

$$= R^{ij}\frac{\partial \phi}{\partial x^j} \tag{14}$$

$$= R^{ij}\left(\nabla\phi\right)^j \tag{15}$$

The gradient therefore transforms like a tensor so it is a tensor.

Example 2. What about the square of the gradient of a scalar? We have

$$(\nabla\phi)\cdot(\nabla\phi) = \sum_i\left(\frac{\partial \phi}{\partial x^i}\right)^2 \tag{16}$$

Rotating the coordinates gives

$$\left(\nabla'\phi\right)\cdot\left(\nabla'\phi\right) = \sum_i\left(\frac{\partial \phi}{\partial x'^i}\right)^2 \tag{17}$$

$$= R^{ij}\frac{\partial \phi}{\partial x^j}R^{ik}\frac{\partial \phi}{\partial x^k} \tag{18}$$

We can now use the fact that $R^T = R^{-1}$ so $R^{ij} = \left(R^{-1}\right)^{ji}$:

$$\left(\nabla'\phi\right)\cdot\left(\nabla'\phi\right) = \left(R^{-1}\right)^{ji} R^{ik}\frac{\partial\phi}{\partial x^j}\frac{\partial\phi}{\partial x^k} \tag{19}$$

$$= \delta^{jk}\frac{\partial\phi}{\partial x^j}\frac{\partial\phi}{\partial x^k} \tag{20}$$

$$= \sum_j\left(\frac{\partial\phi}{\partial x^j}\right)^2 \tag{21}$$

$$= (\nabla\phi)\cdot(\nabla\phi) \tag{22}$$

Thus $(\nabla\phi)\cdot(\nabla\phi)$ is invariant under rotation, so it's a scalar.

Example 3. The Laplacian of a scalar is also a scalar. The Laplacian is defined as

$$\nabla^2\phi \equiv \sum_i \frac{\partial^2\phi}{\left(\partial x^i\right)^2} \tag{23}$$

$$= \frac{\partial}{\partial x^i}\left(\frac{\partial\phi}{\partial x^i}\right) \tag{24}$$

Under rotation, we have

$$\nabla'^2\phi = \frac{\partial}{\partial x'^i}\left(\frac{\partial\phi}{\partial x'^i}\right) \tag{25}$$

$$= R^{ij}\frac{\partial}{\partial x^j}\left(R^{ik}\frac{\partial\phi}{\partial x^k}\right) \tag{26}$$

$$= \left(R^{-1}\right)^{ji} R^{ik}\frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^k}\right) \tag{27}$$

$$= \delta^{jk}\frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^k}\right) \tag{28}$$

$$= \frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^j}\right) \tag{29}$$

$$= \nabla^2\phi \tag{30}$$

Note that the rotation matrix R is a constant with respect to $\frac{\partial}{\partial x^j}$ since we're considering a fixed rotation. The derivative measures how the scalar field ϕ varies as we move from one point to another, whereas R relates one set of fixed coordinates to another set of fixed coordinates.

Thus $\nabla^2\phi$ is invariant under rotation, so it's a scalar.

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