DECOMPOSITION OF A RANK 2 TENSOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problem 2.

Post date: 8 Apr 2020.

In Zee's book, he defines a tensor as "something that transforms like a tensor". For a tensor with N indices, under a rotation specified by the matrix R, the transformation of the tensor is given by multiplying the original tensor by one copy of R for each index. For a 2-index tensor, for example

$$T'^{ij} = R^{ik} R^{j\ell} T^{k\ell} \tag{1}$$

Another way of looking at this transformation is to think of each component T^{ij} of the tensor as a separate object in its own right. We can then arrange these objects in a column matrix (I'm avoiding calling this column matrix a 'vector' since, as Zee points out, vectors have a specific transformation property that this column matrix doesn't have, namely that it must transform under a rotation by a multiplication by a single instance of a rotation matrix R). For 3-d, for example, we have the 9-component matrix

$$\mathcal{T} = \begin{bmatrix} T^{11} \\ T^{12} \\ \vdots \\ T^{33} \end{bmatrix} \tag{2}$$

Under a rotation, we see from 1 that the transformed tensor component T'^{ij} is a *linear* combination of the original components $T^{k\ell}$, where the coefficients of this linear transformation are found from the elements of the rotation matrix R. This means that we could define a matrix $\mathcal{D}$ which, in 3-d, is of size 9×9 and whose elements are composed of combinations of the elements of R. That is

$$\mathcal{T}' = \mathcal{D}\mathcal{T} \tag{3}$$

For example

$$\left(\mathcal{T}'\right)^{11} = T'^{11} = \mathcal{D}^{ij}\mathcal{T}^{j} \tag{4}$$

where the index j is summed from $j = 1$ to $j = 9$. We can read off the first row of $\mathcal{D}$ from 1, as this is the row of $\mathcal{D}$ which provides the coefficients for producing the transformed component T'^{11}.

$$\mathcal{D}^{1j} = \left[\begin{array}{cccccc} R^{11}R^{11} & R^{11}R^{12} & R^{11}R^{13} & R^{12}R^{11} & \ldots & R^{13}R^{13} \end{array}\right] \tag{5}$$

For a general rank 2 tensor (a tensor having 2 indices), there aren't any pre-defined symmetries, so all the elements are independent of each other. As such, a transformed component T'^{ij} could have a contribution from all 9 of the original components $T^{k\ell}$. However, it's possible to create linear combinations of the original T^{ij}s such that a subset of these linear combinations transform into each other.

One such subset contains the antisymmetric combinations

$$A^{ij} \equiv T^{ij} - T^{ji} \tag{6}$$

Zee shows that an antisymmetric component transforms as

$$A'^{ij} = R^{ik}R^{j\ell}A^{k\ell} \tag{7}$$

That is, antisymmetric components transform as linear combinations of *only* other antisymmetric components. In 3-d the index i in A^{ij} can have 3 values, while j can have only 2 (since A^{ii} is always zero by definition, we don't count it). Also, since we're after only components that are linearly independent of each other, we don't count A^{ji} once we've counted A^{ij}, so there are a total of $\frac{3\times 2}{2} = 3$ independent A^{ij}. In D dimensions, there are $\frac{1}{2}D(D-1)$ independent antisymmetric combinations. These components transform entirely within their own private subset.

We can also define a set S^{ij} of symmetric components as

$$S^{ij} \equiv T^{ij} + T^{ji} \tag{8}$$

These components transform as follows:

$$\begin{aligned} S'^{ij} &= T'^{ij} + T'^{ji} && (9)\\ &= R^{ik}R^{j\ell}T^{k\ell} + R^{jk}R^{i\ell}T^{k\ell} && (10)\\ &= R^{ik}R^{j\ell}T^{k\ell} + R^{j\ell}R^{ik}T^{\ell k} && (11)\\ &= R^{ik}R^{j\ell}\left(T^{k\ell} + T^{\ell k}\right) && (12)\\ &= R^{ik}R^{j\ell}S^{k\ell} && (13) \end{aligned}$$

In the third line, we swapped the dummy summed indices k and ℓ. Thus the symmetric combinations also transform within their own subset. There are $\frac{1}{2}D(D-1)$ plus the D diagonal components S^{ii} (no sum) which are, in general, non-zero, for a total of $\frac{1}{2}D(D+1)$ symmetric components. Together the antisymmetric and symmetric components contain all $\frac{1}{2}D(D-1)+\frac{1}{2}D(D+1)=D^2$ independent linear combinations in the original tensor T^{ij}. This means that any of the original tensor components can be written as a combination of the A^{ij} and S^{ij} as

$$T^{ij}=\frac{1}{2}\left(A^{ij}+S^{ij}\right) \tag{14}$$

This decomposition also works for diagonal elements since $A^{ii}=0$ and $S^{ii}=2T^{ii}$ (no sums).

If we write the original tensor in terms of the A^{ij} and S^{ij}, then (in 3-d) the matrix $\mathcal{D}$ decomposes into a block diagonal matrix with a 3×3 block for the A^{ij} and a 6×6 block for the S^{ij}. That is, the transformation equation becomes

$$T'^{ij}=\frac{1}{2}\left(A'^{ij}+S'^{ij}\right) \tag{15}$$
$$=\frac{1}{2}R^{ik}R^{j\ell}\left(A^{k\ell}+S^{k\ell}\right) \tag{16}$$

For example, if we want T'^{32} we have

$$T'^{32}=\frac{1}{2}\left(A'^{32}+S'^{32}\right) \tag{17}$$
$$=\frac{1}{2}\left(-A'^{23}+S'^{23}\right) \tag{18}$$
$$=\frac{1}{2}R^{2k}R^{3\ell}\left(-A^{k\ell}+S^{k\ell}\right) \tag{19}$$

The sums over $A^{k\ell}$ and $S^{k\ell}$ can now be worked out using the symmetry properties of these elements. For $A^{k\ell}$ we have

$$-R^{2k}R^{3\ell}A^{k\ell}=-R^{21}R^{32}A^{12}-R^{21}R^{33}A^{13}-R^{22}R^{31}A^{21}- R^{22}R^{33}A^{23}-R^{23}R^{31}A^{31}-R^{23}R^{32}A^{32} \tag{20}$$
$$=\left(R^{22}R^{31}-R^{21}R^{32}\right)A^{12}+\left(R^{23}R^{31}-R^{21}R^{33}\right)A^{13}+ \left(R^{23}R^{32}-R^{22}R^{33}\right)A^{23} \tag{21}$$

Thus the third row of the 3×3 block in the matrix $\mathcal{D}$ (which is used to calculate A'^{23}) is

$$\left[\ \left(R^{21}R^{32}-R^{22}R^{31}\right)\quad\left(R^{21}R^{33}-R^{23}R^{31}\right)\quad\left(R^{22}R^{33}-R^{23}R^{32}\right)\ \right] \tag{22}$$

We could do a similar calculation for S^{ij} except this time we'd get 6 terms in the transformation.

In fact the symmetric part of $\mathcal{D}$ can be decomposed further by observing that the trace of the symmetric submatrix is invariant under rotation, as Zee shows in his equation 6 (sum implied over i):

$$S'^{ii} = S^{ii} \tag{23}$$

Therefore the 6×6 matrix breaks into a 1×1 matrix and a 5×5 matrix. Zee shows that the components of the 5×5 block (or $D-1\times D-1$ in the D-dimensional case) are given by

$$\tilde{S}^{ij} = S^{ij} - \delta^{ij}\frac{S^{kk}}{D} \tag{24}$$

Zee gives an example in 3-d showing that the components of $\tilde{S}^{ij}$ do indeed transform into themselves.