VOLUME ELEMENT IS A SCALAR

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problem 3.

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The 3-d infinitesimal volume element d^3x is a scalar. Intuitively, this seems reasonable, since a volume isn't going to change by being rotated. Zee's definition of a tensor is something that transforms like a tensor, which means a transformation by rotation. Zee shows in chapter I.3 that the dot product of two vectors is a scalar, and we've seen that the cross product of two vectors transforms like a vector under rotation. We can use these two facts to demonstrate that the volume element is a scalar.

The volume of a parallelopiped with edges given by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by the scalar triple product

$$V = \left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right| \tag{1}$$

The rectangular volume element d^3x can be thought of as a cube with sides given by vectors $d\vec{x}^1$, $d\vec{x}^2$ and $d\vec{x}^3$, so its volume is

$$d^3x = \left|d\vec{x}^1\cdot\left(d\vec{x}^2\times d\vec{x}^3\right)\right| \tag{2}$$

Since this is the dot product of two vectors, it transforms like a scalar (that is, it's constant under rotation).