# LAPLACE-RUNGE-LENZ VECTOR

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problem 4.

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In the last section of chapter I.4, Zee introduces the Laplace-Runge-Lenz vector in the context of planetary motion. The vector is defined as

I'll revert to using boldface type instead of arrows to indicate vectors as it is easier to type.

$$\mathcal{L}=\boldsymbol{\ell}\times\dot{\boldsymbol{r}}+\frac{\kappa}{r}\boldsymbol{r} \tag{1}$$

where

$$\kappa=GM \tag{2}$$

and $M$ is the mass of the body about which the planet is orbiting (the Sun, in our case). We're taking the mass of the planet to be 1 in this case.

For a central force, the angular momentum $\boldsymbol{\ell}$ is conserved, so $\dot{\boldsymbol{\ell}}=0$. Taking the time derivative of 1 therefore gives

$$\dot{\mathcal{L}}=\boldsymbol{\ell}\times\ddot{\boldsymbol{r}}-\frac{\kappa\dot{r}}{r^{2}}\boldsymbol{r}+\frac{\kappa}{r}\dot{\boldsymbol{r}} \tag{3}$$

For a central, inverse-square force, Newton's law gives us

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \text{Force} &(4)\\ &= -\frac{GM}{r^{3}}\boldsymbol{r} &(5)\\ &= -\frac{\kappa}{r^{3}}\boldsymbol{r} &(6)\end{aligned}$$

The angular momentum for a unit mass is

$$\boldsymbol{\ell}=\boldsymbol{r}\times\boldsymbol{p}=\boldsymbol{r}\times\dot{\boldsymbol{r}} \tag{7}$$

The first term in 3 is therefore

$$\boldsymbol{\ell}\times\ddot{\boldsymbol{r}}=-\frac{\kappa}{r^{3}}(\boldsymbol{r}\times\dot{\boldsymbol{r}})\times\boldsymbol{r} \tag{8}$$

We can use the vector identity

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A} \tag{9}$$

so we get

$$\boldsymbol{\ell}\times\ddot{\boldsymbol{r}} = -\frac{\kappa}{r^3}\left(r^2\dot{\boldsymbol{r}} - (\boldsymbol{r}\cdot\dot{\boldsymbol{r}})\,\boldsymbol{r}\right) \tag{10}$$

At this point, we need to be careful with the notation. The time derivative $\dot{r}$ is *not* the magnitude of the vector $\dot{\boldsymbol{r}}$. The symbol $r$ denotes the magnitude of $\boldsymbol{r}$, that is the distance of the planet from the sun. Its derivative is the rate of change of this distance and does not take into account that $\dot{\boldsymbol{r}}$ also has a transverse component which carries the planet in its orbit. The symbol $\dot{r}$ is therefore the value of the component of $\dot{\boldsymbol{r}}$ that is parallel to $\boldsymbol{r}$. Because the dot product of perpendicular vectors is zero, this means that in 10 the dot product of $\boldsymbol{r}$ with the transverse component of $\dot{\boldsymbol{r}}$ is zero, and we have

$$\boldsymbol{r}\cdot\dot{\boldsymbol{r}} = r\dot{r} \tag{11}$$

We therefore have

$$\boldsymbol{\ell}\times\ddot{\boldsymbol{r}} = -\frac{\kappa}{r}\dot{\boldsymbol{r}} + \frac{\kappa}{r^2}\dot{r}\boldsymbol{r} \tag{12}$$

Substituting this back into 3 we find

$$\dot{\mathcal{L}} = 0 \tag{13}$$

that is, the Laplace-Runge-Lenz vector is a constant of the motion.

The significance of this is that we can look at the situation at perihelion or aphelion (closest or furthest approach to the Sun) at which points the planet's velocity is entirely transverse, so that $\dot{\boldsymbol{r}}$ is perpendicular to $\boldsymbol{r}$. In that case $\boldsymbol{r}\cdot\dot{\boldsymbol{r}} = 0$ and we see from 1, 7 and 9 that

$$\boldsymbol{\ell}\times\dot{\boldsymbol{r}} = (\boldsymbol{r}\times\dot{\boldsymbol{r}})\times\dot{\boldsymbol{r}} \tag{14}$$

$$= (\boldsymbol{r}\cdot\dot{\boldsymbol{r}})\,\dot{\boldsymbol{r}} - (\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\,\boldsymbol{r} \tag{15}$$

$$= -(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\,\boldsymbol{r} \tag{16}$$

Thus

$$\mathcal{L} = \left(\frac{\kappa}{r} - (\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\right)\boldsymbol{r} \tag{17}$$

so that $\mathcal{L}$ is parallel to $\boldsymbol{r}$. If we choose the initial value of $\mathcal{L}$ to be parallel to the perihelion direction, this remains constant so the perihelion doesn't precess and the orbit closes. This isn't true in general relativity.

In the case of Mercury, there is a perihelion shift even in Newtonian mechanics, but this is due to the influence of the other planets in the solar system. The point is that the prediction of this shift from Newtonian theory didn't agree with the actual value, and this is where general relativity solved the problem.