TENSORS: A FEW EXAMPLES

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problems 5 - 7.

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Here are a few examples of tensors.

Example 1. If S^{ij} is a symmetric tensor and A^{ij} is an antisymmetric tensor, then using the summation convention on both indices

$$S^{ij}A^{ij}=-S^{ji}A^{ji}=-S^{ij}A^{ij} \tag{1}$$

In the first step, we used $S^{ij}=S^{ji}$ and $A^{ij}=-A^{ji}$. In the second step we just relabelled the dummy indices. Since this shows that $S^{ij}A^{ij}$ is equal to its negative, it must be zero.

$$S^{ij}A^{ij}=0 \tag{2}$$

Example 2. We now consider a totally antisymmetric tensor T^{ijk} in D dimensions. Being totally antisymmetric means that if we swap any two indices, the tensor component becomes its negative. Thus if any two indices are equal, the component must be zero. If all three indices are different and we do two swaps, we are back where we started. This means that we have D choices for the first index, $D-1$ for the second and $D-2$ for the third. Once we have chosen these index values, we also know the value of any tensor component obtained by permuting these 3 indices. Since there are $3!=6$ ways of permuting 3 different indices, the number of independent components of T^{ijk} is $\frac{1}{3!}D(D-1)(D-2)$. If $D=3$, there is only one independent component which we can take to be T^{123}.

Example 3. For the tensor T^{123} in the previous example, we can show that it transforms as a scalar. Under a rotation, we have

$$T'^{123}=R^{1i}R^{2j}R^{3k}T^{ijk} \tag{3}$$

$$=R^{11}\left(R^{22}R^{33}-R^{23}R^{32}\right)T^{123}+R^{12}\left(R^{23}R^{31}-R^{21}R^{33}\right)T^{231}+ R^{13}\left(R^{21}R^{32}-R^{22}R^{31}\right)T^{312} \tag{4}$$

Here, we've used the antisymmetry property so that $T^{123} = -T^{132}$ and so on. For an antisymmetric tensor, any cyclic permutation of the indices leaves the tensor component unchanged so

$$T^{123} = T^{231} = T^{312} \tag{5}$$

and we have

$$T'^{123} = \left[R^{11}\left(R^{22}R^{33} - R^{23}R^{32}\right) + R^{12}\left(R^{23}R^{31} - R^{21}R^{33}\right) + R^{13}\left(R^{21}R^{32} - R^{22}R^{31}\right)\right]T^{123} \tag{6}$$

We can now use the property that we showed earlier to relate the components of the 3-d rotation matrix:

We could also note that the quantity in square brackets in 6 is $\det R = 1$.

$$\begin{aligned} R^{11} &= R^{22}R^{33} - R^{23}R^{32} \\ R^{12} &= R^{31}R^{23} - R^{21}R^{33} \\ R^{13} &= R^{21}R^{32} - R^{31}R^{22} \end{aligned} \tag{7}$$

and

$$\left(R^{11}\right)^2 + \left(R^{12}\right)^2 + \left(R^{13}\right)^2 = 1 \tag{8}$$

Applying this to 6 we have

$$T'^{123} = \left[\left(R^{11}\right)^2 + \left(R^{12}\right)^2 + \left(R^{13}\right)^2\right]T^{123} \tag{9}$$

$$= T^{123} \tag{10}$$

Thus T^{123} transforms into itself and thus acts like a scalar.