ROTATION MATRIX: A THEOREM ABOUT ITS DETERMINANT

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.4, Problem 9.

Post date: 10 Apr 2020.

In chapter I.4, eqn 13, Zee states the theorem that, for a rotation matrix R^{ij} in D dimensions:

$$\varepsilon^{ijk\ldots n}R^{ip}R^{jq}R^{kr}\ldots R^{ns}=\varepsilon^{pqr\ldots s}\det R=\varepsilon^{pqr\ldots s} \tag{1}$$

where $\varepsilon^{ijk\ldots n}$ is the totally antisymmetric tensor with D indices. The last step follows because a rotation matrix satisfies

$$\det R=1 \tag{2}$$

We can verify this for the cases $D=2$ and $D=3$. For $D=2$, we have

$$\varepsilon^{ij}R^{ip}R^{jq}=R^{1p}R^{2q}-R^{2p}R^{1q} \tag{3}$$

If $p=q$, this is zero. Otherwise, suppose $p=1$ and $q=2$. Then

$$\begin{aligned}\varepsilon^{ij}R^{i1}R^{j2}&=R^{11}R^{22}-R^{21}R^{12} &(4)\\&=\det R &(5)\\&=\varepsilon^{12}\det R &(6)\end{aligned}$$

If $p=2$ and $q=1$, then

$$\begin{aligned}\varepsilon^{ij}R^{i2}R^{j1}&=R^{12}R^{21}-R^{22}R^{11} &(7)\\&=-\det R &(8)\\&=\varepsilon^{21}\det R &(9)\end{aligned}$$

Thus 1 is true for $D=2$.

For $D=3$, we have

$$\begin{aligned}\varepsilon^{ijk}R^{ip}R^{jq}R^{kr} &= R^{1p}R^{2q}R^{3r} - R^{1p}R^{3q}R^{2r} - R^{2p}R^{1q}R^{3r} + \\ &\quad R^{2p}R^{3q}R^{1r} + R^{3p}R^{1q}R^{2r} - R^{3p}R^{2q}R^{1r} \qquad (10)\\ &= R^{1p}\left(R^{2q}R^{3r} - R^{3q}R^{2r}\right) - R^{2p}\left(R^{1q}R^{3r} - R^{3q}R^{1r}\right) + \\ &\quad R^{3p}\left(R^{1q}R^{2r} - R^{2q}R^{1r}\right) \qquad (11)\end{aligned}$$

We can try various combinations of p, q and r. For example, $p = 1$, $q = 2$ and $r = 3$ and we have

$$\varepsilon^{ijk}R^{i1}R^{j2}R^{k3} = \det R \qquad (12)$$

Swapping q and r gives $-\det R$ and so on. In general, 11 is the formula for the determinant expanded about the pth column of R. Swapping any two rows or two columns of a determinant changes its sign (theorem from matrix algebra), so swapping any pair of p, q and r changes the sign of the result. Thus

$$\varepsilon^{ijk}R^{ip}R^{jq}R^{kr} = \varepsilon^{pqr}\det R = \varepsilon^{pqr} \qquad (13)$$