

## ESKIMO MITES AND THEIR METRIC

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 2.

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Zee considers a fictional race of mites that live near the north pole of a unit sphere (radius = 1). They use the north pole as their origin and consider only short distances from this origin. The usual cartesian coordinates in this case become

$$x = \sin \theta \cos \phi \approx \theta \cos \phi \quad (1)$$

$$y = \sin \theta \sin \phi \approx \theta \sin \phi \quad (2)$$

The mites have worked out that their metric, to second order, is given by

$$ds^2 = \left(1 - \frac{y^2}{3}\right) dx^2 + \left(1 - \frac{x^2}{3}\right) dy^2 + \frac{2}{3}xy dx dy + \dots \quad (3)$$

We need to show that this is equivalent to the ordinary metric in spherical coordinates which is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (4)$$

from which we get

$$dx = \cos \phi d\theta - \theta \sin \phi d\phi \quad (5)$$

$$dy = \sin \phi d\theta + \theta \cos \phi d\phi \quad (6)$$

We can solve for  $d\theta$  and  $d\phi$  to get

$$d\theta = \cos \phi dx + \sin \phi dy \quad (7)$$

$$d\phi = \frac{1}{\theta} (\cos \phi dy - \sin \phi dx) \quad (8)$$

Plugging this into 4 and expanding, we have

$$ds^2 = (\cos \phi dx + \sin \phi dy)^2 + \frac{\sin^2 \theta}{\theta^2} (\cos \phi dy - \sin \phi dx)^2 \quad (9)$$

We can now approximate

$$\sin \theta \approx \theta - \frac{\theta^3}{3!} \quad (10)$$

and keep only terms up to  $\theta^2$  in the expansion of  $\frac{\sin^2 \theta}{\theta^2}$ , along with  $\cos^2 \phi + \sin^2 \phi = 1$  to get

$$ds^2 = dx^2 + dy^2 - 2 \frac{\theta^2}{3!} (\cos \phi dy - \sin \phi dx)^2 \quad (11)$$

$$= \left(1 - \frac{1}{3} (\theta \sin \phi)^2\right) dx^2 + \left(1 - \frac{1}{3} (\theta \cos \phi)^2\right) dy^2 + \frac{2}{3} \theta^2 \sin \phi \cos \phi \quad (12)$$

$$= \left(1 - \frac{y^2}{3}\right) dx^2 + \left(1 - \frac{x^2}{3}\right) dy^2 + \frac{2}{3} xy dx dy \quad (13)$$

where we used 1 to get the last line.

We could also do this by starting with 1. We plug 1 and 5 into 3 and expand. I used Maple to do this, as it's somewhat messy, with the result

$$ds^2 = d\theta^2 + \frac{1}{3} (3\theta^2 - \theta^4) d\phi^2 \quad (14)$$

To first order in  $\theta$ ,  $\sin \theta \approx \theta$  and we can ignore the  $\theta^4$  term, so we get 4, which verifies the result.

If you really want to grind through the calculations by hand, here is what Maple gives for intermediate stages in the calculation.

$$\begin{aligned} \left(1 - \frac{y^2}{3}\right) dx^2 &= -\frac{1}{3} (\theta^2 \cos^2 \phi - \theta^2 + 3) \times \\ &\quad (\theta^2 \cos^2 \phi d\phi^2 + 2\theta \cos \phi \sin(\phi) d\theta d\phi - \cos^2 \phi d\theta^2 - \theta^2 d\phi^2) \end{aligned} \quad (15)$$

$$\begin{aligned} \left(1 - \frac{x^2}{3}\right) dy^2 &= -\frac{1}{3} (\theta^2 \cos^2 \phi - 3) \times \\ &\quad (\theta^2 \cos^2 \phi d\phi^2 + 2\theta \cos \phi \sin(\phi) d\theta d\phi - \cos^2 \phi d\theta^2 + d\theta^2) \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{2}{3} xy dx dy &= -\frac{2}{3} \theta^2 \cos(\phi) \sin(\phi) \times \\ &\quad (-2\theta \cos^2 \phi d\phi d\theta + \sin \phi \cos \phi (\theta^2 d\phi^2 - d\theta^2) + \theta d\phi d\theta) \end{aligned} \quad (17)$$

When we add up these three terms and simplify, we get 14.

PINGBACKS

Pingback: [Eskimo mite metric revisited](#)