MERCATOR MAP METRIC

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 3.

Post date: 13 Apr 2020.

For the Mercator projection of the map of the world, the grid coordinates (x,y) on the 2-d map are given in terms of the usual spherical coordinates (θ,ϕ) by

$$x = \frac{W}{2\pi}\phi \tag{1}$$

$$y = -\frac{W}{2\pi}\ln\tan\left(\frac{\theta}{2}\right) \tag{2}$$

Note that θ is not the same as latitude (which is 0 at the equator and $\pm\frac{\pi}{2}$ at the poles), but is the usual spherical angle which is 0 at the north pole, $\frac{\pi}{2}$ at the equator and π at the south pole.

We can relate the usual spherical metric, which is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{3}$$

to the Mercator metric. We'll start by working out dx^2+dy^2. I've used Maple to do some simplifications.

$$dx^2+dy^2 = \frac{W^2}{4\pi^2}\left[d\phi^2 + \frac{1}{4}\frac{\left(1+\tan^2\frac{\theta}{2}\right)^2}{\tan^2\frac{\theta}{2}}\right]d\theta^2 \tag{4}$$

To simplify this, we can use the trig identity

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \tag{5}$$

Using this, we get

$$dx^2+dy^2 = \frac{W^2}{4\pi^2}\left[\frac{d\theta^2}{\sin^2\theta}+d\phi^2\right] \tag{6}$$

$$= \frac{W^2}{4\pi^2\sin^2\theta}\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{7}$$

Comparing this to 3, we see that

$$ds^2 = \Omega^2 \left(dx^2 + dy^2\right) \tag{8}$$

with

$$\Omega = \frac{2\pi \sin\theta}{W} \tag{9}$$

To get Zee's answer at the back of the book, we use 2 in the form

$$\tan\frac{\theta}{2} = e^{-2\pi y/W} \tag{10}$$

and another trig identity

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \tag{11}$$

Using these, we have (with $s \equiv \sin\theta$, $c \equiv \cos\theta$ and $e \equiv e^{-2\pi y/W}$):

$$s - e = ce \tag{12}$$

$$(s-e)^2 = \left(1-s^2\right) e^2 \tag{13}$$

$$s^2\left(1+e^2\right) - 2se = 0 \tag{14}$$

Discounting the $s = \sin\theta = 0$ solution, we have, restoring the original variables

$$\sin\theta = \frac{2e^{-2\pi y/W}}{1+e^{-4\pi y/W}} \tag{15}$$

$$= \frac{2}{e^{2\pi y/W} + e^{-2\pi y/W}} \tag{16}$$

$$= \frac{1}{\cosh\frac{2\pi y}{W}} \tag{17}$$

Plugging this back into 9 we have

$$\Omega = \frac{2\pi}{W\cosh\frac{2\pi y}{W}} \tag{18}$$