BOYER-LINDQUIST COORDINATES AND CURVATURE OF SPACE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 4.

Post date: 13 Apr 2020.

We are given the metric (note that the equation in Zee's book for exercise I.5.4 is missing a factor of $\sin^2\theta$):

$$ds^2 = \frac{\rho^2}{\rho^2 + a^2\sin^2\theta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 \tag{1}$$

This is the same metric as Zee discusses in Appendix 2, with

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2}$$

The relation to the usual cartesian coordinates is given by

$$x = f(r)\sin\theta\cos\phi \tag{3}$$
$$y = f(r)\sin\theta\sin\phi \tag{4}$$
$$z = r\cos\theta \tag{5}$$

with $f^2(r) = r^2 + a^2$. That is, we have

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi \tag{6}$$
$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \tag{7}$$
$$z = r\cos\theta \tag{8}$$

Thus if we set $a = 0$ we recover the usual spherical coordinates.

For $a \neq 0$, if we set $r = 0$ then $z = 0$ so we're restricted to the xy plane. However, we still have a range of values possible for x and y, so $r = 0$ describes a disc rather than a single point. For a fixed value of θ, the point (x,y) is a point on this disc with radius $\sin\theta$. As we vary ϕ, this point describes a circle around the origin. Thus as $\sin\theta$ increases from 0 to 1, we have successively larger circles up to a circle with radius a.

To investigate the curvature of this metric, we can follow the procedure in Zee's Appendix 1, where we evaluate the curvature given by R, defined as

$$R = \lim_{\text{radius}\to 0} \frac{6}{\text{radius}^2}\left(1 - \frac{\text{circumference}}{2\pi\text{radius}}\right) \tag{9}$$

where 'circumference' and 'radius' refer to a circle drawn around the point at which we wish to find the curvature.

Consider the origin in our current example. The distance from the origin to a circle of radius $\sin\epsilon$ is the distance measured along a curve of constant θ (since θ describes the radius as explained above). So

$$\text{radius} = \int_0^\epsilon \rho(r=0,\theta)\,d\theta = a\int_0^\epsilon \cos\theta\,d\theta \tag{10}$$

$$= a\sin\epsilon \tag{11}$$

The circumference of this circle is the length of the curve with radius $a\sin\epsilon$ as ϕ goes from 0 to 2π. That is

$$\text{circumference} = \int_0^{2\pi} a\sin\epsilon\,d\phi = 2\pi a\sin\epsilon \tag{12}$$

The curvature from 9 is then

$$R = \lim_{\epsilon\to 0}\frac{6}{a^2\sin^2\epsilon}\left(1-\frac{2\pi a\sin\epsilon}{2\pi a\sin\epsilon}\right) = 0 \tag{13}$$

Thus the space is flat at the origin.

To study lines of fixed θ and ϕ (I'm assuming he means having both these fixed at the same time), we can play with the defining equations 7. We have

$$r = \frac{z}{\cos\theta} \tag{14}$$

so

$$x^2 = \left(\frac{z^2}{\cos^2\theta} + a^2\right)\sin^2\theta\cos^2\phi \tag{15}$$

$$y^2 = \left(\frac{z^2}{\cos^2\theta} + a^2\right)\sin^2\theta\sin^2\phi \tag{16}$$

Therefore

$$\frac{x^2}{\cos^2\phi}-\frac{y^2}{\sin^2\phi}=0 \tag{17}$$

$$y=\pm x\tan\phi \tag{18}$$

Thus for a fixed value of ϕ, the plot of y versus x is a straight line.

We also have

$$\frac{z^2}{\cos^2\theta}=\frac{x^2+y^2}{\sin^2\theta}-a^2 \tag{19}$$

$$=\frac{x^2}{\sin^2\theta}\left(1+\tan^2\phi\right)-a^2 \tag{20}$$

$$=\frac{x^2}{\sin^2\theta\cos^2\phi}-a^2 \tag{21}$$

Rearranging, we have

$$\frac{x^2}{\sin^2\theta\cos^2\phi}-\frac{z^2}{\cos^2\theta}=a^2 \tag{22}$$

For fixed θ, ϕ and a this is the equation of a hyperbola, with asymptotes given by

$$x=\pm\frac{\tan\theta}{\cos\phi}z \tag{23}$$

PINGBACKS

Pingback: Curvature of 2-dimensional space
Pingback: Curvature of a sphere