CURVATURE OF A SPHERE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 6.

Post date: 16 Apr 2020.

The metric on a unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1}$$

We can calculate the curvature at the north pole by using the method from earlier. A small circle of radius ϵ centred at the north pole has a radius that is a distance obtained by increasing the polar angle θ from 0 at the pole to ϵ:

$$\text{radius} = \int_0^{\epsilon} d\theta = \epsilon \tag{2}$$

The circumference of this circle is

$$\text{circum} = \int_0^{2\pi} \sin\theta \; d\phi \tag{3}$$

$$= \sin\epsilon \int_0^{2\pi} \; d\phi \tag{4}$$

$$= 2\pi \sin\epsilon \tag{5}$$

The curvature is obtained from the formula

$$R = \lim_{\text{radius}\to 0} \frac{6}{\text{radius}^2}\left(1 - \frac{\text{circumference}}{2\pi\text{radius}}\right) \tag{6}$$

which in this case is

$$R = \lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{2\pi\sin\epsilon}{2\pi\epsilon}\right) \tag{7}$$

$$= \lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{\sin\epsilon}{\epsilon}\right) \tag{8}$$

By expanding the sine in a series we have

$$\frac{\sin\epsilon}{\epsilon} = \frac{1}{\epsilon}\left(\epsilon - \frac{\epsilon^3}{3!} + \dots\right) \tag{9}$$

$$= 1 - \frac{\epsilon^2}{6} + \dots \tag{10}$$

Inserting this into 8 we have

$$R = +1 \tag{11}$$

Since all points on a sphere are equivalent, this is the curvature of a unit sphere at any point.